ZERO-DIVISORS OF SEMIGROUP MODULES

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ABSTRACT. Let M be an R-module and S a semigroup. Our goal is to discuss zero-divisors of the semigroup module $M[S]$. Particularly we show that if M is an R-module and S a commutative, cancellative and torsion-free monoid, then the $R[S]$-module $M[S]$ has few zero-divisors of size n if and only if the R-module M has few zero-divisors of size n and Property (A).

1. INTRODUCTION

Let S be a commutative semigroup and M be an R-module. One can define the semigroup module $M[S]$ as an $R[S]$-module constructed from the semigroup S and the R-module M similar to the standard definition of semigroup rings. Obviously similar to semigroup rings, the zero-divisors of the semigroup module $M[S]$ are interesting to investigate ([6, p. 82] and [12]).

We write each element of $g \in M[S]$ as "polynomials" $g = m_1X^{s_1} + m_2X^{s_2} + \cdots + m_nX^{s_n}$, where $m_1, \ldots, m_n \in M$ and $s_1, \ldots, s_n$ are distinct elements of S and this representation of g is called the canonical form of g. For $g = m_1X^{s_1} + m_2X^{s_2} + \cdots + m_nX^{s_n}$, we define the content of g to be the R-submodule of M generated by the coefficients of g.

Northcott gave a nice generalization of *Dedekind-Mertens Lemma* as follows: if S is a commutative, cancellative and torsion-free monoid and M is an R-module, then for all $f \in R[S]$ and $g \in M[S]$, there exists a natural number k such that $c(f)^k c(g) = c(f)^{k-1} c(fg)$ ([16]). Dedekind-Mertens Lemma has different versions with various applications ([1], [2], [3], [8], [9], [15], [18], [19], and [20]). One of its interesting consequences is McCoy's Theorem on zero-divisors ([6, p. 96] and [14]): If M is a nonzero R-module and S is a commutative, cancellative and torsion-free monoid, then for all $f \in R[S]$ and $g \in M[S] - \{0\}$, if $fg = 0$, then there exists an $m \in M - \{0\}$ such that $f \cdot m = 0$.

An R-module M is said to have *few zero-divisors of size n*, if $Z_R(M)$ is a finite union of n prime ideals $\mathbf{p}_1, \ldots, \mathbf{p}_n$ of R such that $\mathbf{p}_i \nsubseteq \mathbf{p}_j$ for all $i \neq j$. Also note that an R-module M has *Property (A)*, if each finitely generated ideal $I \subseteq Z_R(M)$ has a nonzero annihilator in M. We use McCoy's Theorem to prove that if M is an R-module and S a commutative, cancellative and torsion-free monoid, then the $R[S]$-module $M[S]$ has few zero-divisors of size n, if and only if the R-module M has few zero-divisors of size n and Property (A).

In this paper all rings are commutative with identity and all modules are unital[1]. Unless otherwise stated, our notation and terminology will follow as closely as possible that of Gilmer [6].

2. Zero-Divisors of Semigroup Modules

Let us recall that if R is a ring and $f = a_0 + a_1X + \cdots + a_nX^n$ is a polynomial on the ring R, then content of f is defined as the R-ideal, generated by the coefficients of f, i.e. $c(f) = (a_0, a_1, \ldots, a_n)$. The content of an element of a semigroup module is a natural generalization of the content of a polynomial as follows:

Definition 1. Let M be an R-module and S be a commutative semigroup. Let $g \in M[S]$ and put $g = m_1X^{s_1} + m_2X^{s_2} + \cdots + m_nX^{s_n}$, where $m_1, \ldots, m_n \in M$ and $s_1, \ldots, s_n \in S$. We define the content of g to be the R-submodule of M generated by the coefficients of g, i.e. $c(g) = (m_1, \ldots, m_n)$.

Theorem 2. *Let S be a commutative monoid and M be a nonzero R-module. Then the following statements are equivalent:*

1. *S is a cancellative and torsion-free monoid.*
2. *For all $f \in R[S]$ and $g \in M[S]$, there is a natural number k such that $c(f)^k c(g) = c(f)^{k-1} c(fg)$.*
3. *(McCoy's Property) For all $f \in R[S]$ and $g \in M[S] - \{0\}$, if $fg = 0$, then there exists an $m \in M - \{0\}$ such that $f \cdot m = 0$.*
4. *For all $f \in R[S]$, $\operatorname{Ann}_M(c(f)) = 0$ if and only if $f \notin Z_{R[S]}(M[S])$.*

Proof. $(1) \to (2)$ has been proved in [16].

For $(2) \to (3)$, assume that $f \in R[S]$ and $g \in M[S] - \{0\}$, such that $fg = 0$. So there exists a natural number k such that $c(f)^k c(g) = c(f)^{k-1} c(fg) = (0)$. Take t the smallest natural number such that $c(f)^t c(g) = (0)$ and choose m a nonzero element of $c(f)^{t-1} c(g)$. It is easy to check that $f \cdot m = 0$.

For $(3) \to (1)$, we prove that if S is not cancellative or not torsion-free then (1) cannot hold. For the moment, suppose that S is not cancellative, so there exist $s, t, u \in S$ such that $s + t = s + u$ while $t \neq u$. Put $f = X^s$ and $g = (qX^t - qX^u)$, where q is a nonzero element of M. Then obviously $fg = 0$, while $f \cdot m \neq 0$ for all $m \in M - \{0\}$. Finally suppose that S is cancellative but not torsion-free. Let $s, t \in S$ be such that $s \neq t$, while $ns = nt$ for some natural n. Choose the natural number k minimal so that $ks = kt$. Then we have $0 = qX^{ks} - qX^{kt} = (\sum_{i=0}^{k-1} X^{(k-i-1)s+it})(qX^s - qX^t)$, where q is a nonzero element of M.

Since S is cancellative, the choice of k implies that $(k - i_1 - 1)s + i_1t \neq (k - i_2 - 1)s + i_2t$ for $0 \leq i_1 < i_2 \leq k - 1$. Therefore $\sum_{i=0}^{k-1} X^{(k-i-1)s+it} \neq 0$, and this completes the proof. $(3) \leftrightarrow (4)$ is obvious. □

Corollary 3. *Let M be an R-module and S be a commutative, cancellative and torsion-free monoid. Then the following statements hold:*

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(1) *R is a domain if and only if $R[S]$ is a domain.*
(2) *If $\mathbf{p}$ is a prime ideal of R, then $\mathbf{p}[S]$ is a prime ideal of $R[S]$.*
(3) *If $\mathbf{p}$ is in $\mathrm{Ass}_R(M)$, then $\mathbf{p}[S]$ is in $\mathrm{Ass}_{R[S]}(M[S])$.*

Definition 4. Let M be an R-module and P be a proper R-submodule of M. P is said to be a *prime submodule* (*primary submodule*) of M, if $rx \in P$ implies $x \in P$ or $rM \subseteq P$ (there exists a natural number n such that $r^n M \subseteq P$), for each $r \in R$ and $x \in M$.

Corollary 5. *Let M be an R-module and S be a commutative, cancellative and torsion-free monoid. Then the following statements hold:*

(1) (0) *is a prime (primary) submodule of M if and only if* (0) *is a prime (primary) submodule of $M[S]$.*
(2) *If P is a prime (primary) submodule of M, then $P[S]$ is a prime (primary) submodule of $M[S]$.*

In [5], it has been defined that a ring R has *few zero-divisors*, if $Z(R)$ is a finite union of prime ideals. We give the following definition and prove some interesting results about zero-divisors of semigroup modules. Modules having (very) few zero-divisors, introduced in [15], have also some interesting homological properties [17].

Definition 6. An R-module M has *very few zero-divisors*, if $Z_R(M)$ is a finite union of prime ideals in $\mathrm{Ass}_R(M)$.

Remark 7. *Examples of modules having very few zero-divisors.* If R is a Noetherian ring and M is an R-module such that $\mathrm{Ass}_R(M)$ is finite, then obviously M has very few zero-divisors. For example $\mathrm{Ass}_R(M)$ is finite if M is a finitely generated R-module [13, p. 55]. Also if R is a Noetherian quasi-local ring and M is a balanced big Cohen-Macaulay R-module, then $\mathrm{Ass}_R(M)$ is finite [4, Proposition 8.5.5, p. 344].

Remark 8. Let R be a ring and consider the following three conditions on R:

(1) R is a Noetherian ring.
(2) R has very few zero-divisors.
(3) R has few zero-divisors.

Then, $(1) \to (2) \to (3)$ and none of the implications are reversible.

Proof. For $(1) \to (2)$ use [13, p. 55]. It is obvious that $(2) \to (3)$.

Suppose k is a field, $A = k[X_1, X_2, X_3, \ldots, X_n, \ldots]$ and $\mathbf{m} = (X_1, X_2, X_3, \ldots, X_n, \ldots)$ and at last $\mathbf{a} = (X_1^2, X_2^2, X_3^2, \ldots, X_n^2, \ldots)$. Since A is a domain, A has very few zero-divisors while it is not a Noetherian ring. Also consider the ring $R = A/\mathbf{a}$. It is easy to check that R is a quasi-local ring with the only prime ideal $\mathbf{m}/\mathbf{a}$ and $Z(R) = \mathbf{m}/\mathbf{a}$ and finally $\mathbf{m}/\mathbf{a} \notin \mathrm{Ass}_R(R)$. Note that $\mathrm{Ass}_R(R) = \emptyset$ [15]. $\square$

Theorem 9. *Let M be an R-module and S a commutative, cancellative and torsion-free monoid. Then the $R[S]$-module $M[S]$ has very few zero-divisors, if and only if the R-module M has very few zero-divisors.*

Proof. $(\leftarrow)$: Let $Z_R(M) = \mathbf{p}_1 \cup \mathbf{p}_2 \cup \cdots \cup \mathbf{p}_n$, where $\mathbf{p}_i \in \mathrm{Ass}_R(M)$ for all $1 \leq i \leq n$. We will show that $Z_{R[S]}(M[S]) = \mathbf{p}_1[S] \cup \mathbf{p}_2[S] \cup \cdots \cup \mathbf{p}_n[S]$. Let $f \in Z_{R[S]}(M[S])$, so there exists an $m \in M - \{0\}$ such that $f \cdot m = 0$ and so $c(f) \cdot m = (0)$. Therefore $c(f) \subseteq Z_R(M)$ and this means that $c(f) \subseteq \mathbf{p}_1 \cup \mathbf{p}_2 \cup \cdots \cup \mathbf{p}_n$ and according to the Prime Avoidance Theorem, we have $c(f) \subseteq \mathbf{p}_i$, for some $1 \leq i \leq n$ and therefore $f \in \mathbf{p}_i[S]$. Now let $f \in \mathbf{p}_1[S] \cup \mathbf{p}_2[S] \cup \cdots \cup \mathbf{p}_n[S]$, so there exists an i such that $f \in \mathbf{p}_i[S]$, so $c(f) \subseteq \mathbf{p}_i$ and $c(f)$ has a nonzero annihilator in M and this means that f is a zero-divisor of $M[S]$. Note that by Corollary 3, $\mathbf{p}_i[S] \in \mathrm{Ass}_{R[S]}(M[S])$ for all $1 \leq i \leq n$.

($\rightarrow$): Let $Z_{R[S]}(M[S]) = \cup_{i=1}^n Q_i$, where $Q_i \in \mathrm{Ass}_{R[S]}(M[S])$ for all $1 \leq i \leq n$. Therefore $Z_R(M) = \cup_{i=1}^n (Q_i \cap R)$. Without loss of generality, we can assume that $Q_i \cap R \nsubseteq Q_j \cap R$ for all $i \neq j$. Now we prove that $Q_i \cap R \in \mathrm{Ass}_R(M)$ for all $1 \leq i \leq n$. Consider $g \in M[S]$ such that $Q_i = \mathrm{Ann}(g)$ and $g = m_1 X^{s_1} + m_2 X^{s_2} + \cdots + m_n X^{s_n}$, where $m_1, \ldots, m_n \in M$ and $s_1, \ldots, s_n \in S$. It is easy to see that $Q_i \cap R = \mathrm{Ann}(c(g)) \subseteq \mathrm{Ann}(m_1) \subseteq Z_R(M)$ and by the Prime Avoidance Theorem, $Q_1 \cap R = \mathrm{Ann}(m_1)$. $\square$

In [11], it has been defined that a ring R has *Property (A)*, if each finitely generated ideal $I \subseteq Z(R)$ has a nonzero annihilator. We give the following definition:

Definition 10. An R-module M has *Property (A)*, if each finitely generated ideal $I \subseteq Z_R(M)$ has a nonzero annihilator in M.

Remark 11. If the R-module M has very few zero-divisors, then M has Property (A).

Theorem 12. *Let S be a commutative, cancellative and torsion-free monoid and M be an R-module. The following statements are equivalent:*

(1) *The R-module M has Property (A).*
(2) *For all $f \in R[S]$, f is $M[S]$-regular if and only if $c(f)$ is M-regular.*

Proof. $(1) \rightarrow (2)$: Let the R-module M have Property (A). If $f \in R[S]$ is $M[S]$-regular, then $f \cdot m \neq 0$ for all nonzero $m \in M$ and so $c(f) \cdot m \neq (0)$ for all nonzero $m \in M$ and according to the definition of Property (A), $c(f) \nsubseteq Z_R(M)$. This means that $c(f)$ is M-regular. Now let $c(f)$ be M-regular, so $c(f) \nsubseteq Z_R(M)$ and this means that $c(f) \cdot m \neq (0)$ for all nonzero $m \in M$ and hence $f \cdot m \neq 0$ for all nonzero $m \in M$. Since S is a commutative, cancellative and torsion-free monoid, f is not a zero-divisor of $M[S]$, i.e. f is $M[S]$-regular.

$(2) \rightarrow (1)$: Let I be a finitely generated ideal of R such that $I \subseteq Z_R(M)$. Then there exists an $f \in R[S]$ such that $c(f) = I$. But $c(f)$ is not M-regular, therefore according to our assumption, f is not $M[S]$-regular. Therefore there exists a nonzero $m \in M$ such that $f \cdot m = 0$ and this means that $I \cdot m = (0)$, i.e. I has a nonzero annihilator in M. $\square$

Let, for the moment, M be an R-module such that the set $Z_R(M)$ of zero-divisors of M is a finite union of prime ideals. One can consider $Z_R(M) = \cup_{i=1}^n \mathbf{p}_i$ such that $\mathbf{p}_i \nsubseteq \cup_{j=1 \wedge j \neq i}^n \mathbf{p}_j$ for all $1 \leq i \leq n$. Obviously we have $\mathbf{p}_i \nsubseteq \mathbf{p}_j$ for all $i \neq j$. Also, it is easy to check that, if $Z_R(M) = \cup_{i=1}^n \mathbf{p}_i$ and $Z_R(M) = \cup_{k=1}^m \mathbf{q}_k$ such that $\mathbf{p}_i \nsubseteq \mathbf{p}_j$ for all $i \neq j$ and $\mathbf{q}_k \nsubseteq \mathbf{q}_l$ for all $k \neq l$, then $m = n$ and $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\} = \{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$, i.e. these prime ideals are uniquely determined (Use the Prime Avoidance Theorem). This is the base for the following definition:

Definition 13. An R-module M is said to have *few zero-divisors of size n*, if $Z_R(M)$ is a finite union of n prime ideals $\mathbf{p}_1, \ldots, \mathbf{p}_n$ of R such that $\mathbf{p}_i \nsubseteq \mathbf{p}_j$ for all $i \neq j$.

Theorem 14. *Let M be an R-module and S a commutative, cancellative and torsion-free monoid. Then the $R[S]$-module $M[S]$ has few zero-divisors of size n, if and only if the R-module M has few zero-divisors of size n and Property (A).*

Proof. $(\leftarrow)$: By considering the R-module M having Property (A), similar to the proof of Theorem 9, we have if $Z_R(M) = \cup_{i=1}^n \mathbf{p}_i$, then $Z_{R[S]}(M[S]) = \cup_{i=1}^n \mathbf{p}_i[S]$. Also it is obvious that $\mathbf{p}_i[S] \subseteq \mathbf{p}_j[S]$ if and only if $\mathbf{p}_i \subseteq \mathbf{p}_j$ for all $1 \leq i, j \leq n$. These two imply that the $R[S]$-module $M[S]$ has few zero-divisors of size n.

$(\rightarrow)$: Note that $Z_R(M) \subseteq Z_{R[S]}(M[S])$. It is easy to check that if $Z_{R[S]}(M[S]) = \cup_{i=1}^n Q_i$, where Q_i are prime ideals of $R[S]$ for all $1 \leq i \leq n$, then $Z_R(M) = \cup_{i=1}^n (Q_i \cap R)$. Now we prove that the R-module M has Property (A). Let $I \subseteq Z_R(M)$ be a finite ideal of R. Choose $f \in R[S]$ such that $I = c(f)$. So $c(f) \subseteq Z_R(M)$ and obviously $f \in Z_{R[S]}(M[S])$ and according to McCoy's property, there exists a nonzero $m \in M$ such that $f \cdot m = 0$. This means that $I \cdot m = 0$ and I has a nonzero annihilator in M. Consider that by a similar discussion in $(\leftarrow)$, the R-module M has few zero-divisors obviously not less than size n and this completes the proof. $\square$

An R-module M is said to be *primal*, if $Z_R(M)$ is an ideal of R [5]. It is easy to check that if $Z_R(M)$ is an ideal of R, then it is a prime ideal and therefore the R-module M is primal if and only if M has few zero-divisors of size one.

Corollary 15. *Let M be an R-module and S a commutative, cancellative and torsion-free monoid. Then the $R[S]$-module $M[S]$ is primal, if and only if the R-module M is primal and has Property (A).*

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