CONTENT ALGEBRAS AND ZERO-DIVISORS

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Contents

Abstract

This thesis concerns two topics. The first topic, that is related to the Dedekind-Mertens Lemma, the notion of the so-called content algebra, is discussed in chapter 2. Let R be a commutative ring with identity and M be a unitary R-module and c the function from M to the ideals of R defined by $c(x) = \cap\{I\colon I \text{ is an ideal of } R \text{ and } x \in IM\}$. M is said to be a *content* R-module if $x \in c(x)M$, for all $x \in M$. The R-algebra B is called a *content* R-algebra, if it is a faithfully flat and content R-module and it satisfies the Dedekind-Mertens content formula. In chapter 2, it is proved that in content extensions, minimal primes extend to minimal primes, and zero-divisors of a content algebra over a ring which has Property (A) or whose set of zero-divisors is a finite union of prime ideals are discussed. The preservation of diameter of zero-divisor graph under content extensions is also examined. Gaussian and Armendariz algebras and localization of content algebras at the multiplicatively closed set $S' = \{f \in B\colon c(f) = R\}$ are considered as well.

In chapter 3, the second topic of the thesis, that is about the grade of the zero-divisor modules, is discussed. Let R be a commutative ring, I a finitely generated ideal of R, and M a zero-divisor R-module. It is shown that the M-grade of I defined by the Koszul complex is consistent with the definition of M-grade of I defined by the length of maximal M-sequences in I.

Chapter 1 is a preliminarily chapter and dedicated to the introduction of content modules and also locally Nakayama modules.

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Introduction

Throughout this thesis all rings are commutative with unit and all modules are assumed to be unitary. Let R be a commutative ring with identity and X be an indeterminate over R. The *content* $c(f)$ of a polynomial $f \in R[X]$ is the ideal of R generated by the coefficients of f. In other words, if $f = a_0 + a_1X + \cdots + a_nX^n$, where $a_i \in R$ for all $0 \leq i \leq n$ and $n \geq 0$, then $c(f) = (a_0, a_1, \ldots, a_n)$. From the definition of the content of a polynomial, it is obvious that $c(f) = (0) \Leftrightarrow f = 0$ for all $f \in R[X]$. Let, for the moment, $f \in R[X]$ and I be an ideal of R. Consider $f + I[X] \in R[X]/I[X]$ and note that $R[X]/I[X] \cong (R/I)[X]$. It is easy to check that $c(f + I[X]) = c(f) + I$. Also we know that $c(f + I[X]) = (I)$ iff $f + I[X] = I[X]$ (Note that *iff* always stands for if and only if). This means that $c(f) \subseteq I$ iff $f \in I[X]$ that obviously implies that $f \in c(f)[X]$ for all $f \in R[X]$.

Now let M be a unitary R-module and define d a function from M to the ideals of R with this property that $x \in IM$ iff $d(x) \subseteq I$ for all $x \in M$ and ideals I of R. From the definition of d, it is obvious that $d(x)$ is a subset of the intersection of all ideals I of R such that $x \in IM$. Let us define the *content function*, c from M to the ideals of R defined by

$$c(x) = \bigcap \{I \colon I \text{ is an ideal of } R \text{ and } x \in IM\}.$$

Therefore $d(x) \subseteq c(x)$. On the other hand, $x \in d(x)M$ and according to the definition of the content function c, we have $c(x) \subseteq d(x)$ and therefore $d = c$. Note that $x \in c(x)M$ for all $x \in M$. This can be the inspiration of the definition of content modules in [**OR**]: The unitary R-module M is called a *content R-module* if $x \in c(x)M$, for all $x \in M$. The class of content modules are themselves considerable and interesting in the field of module theory. In chapter 1, we introduce content modules and mention some of their basic properties that we need in the rest of the thesis for the convenience of the reader.

From the definition of the content of a polynomial, it is obvious that $c(rf) = rc(f)$ for all $f \in R[X]$ and $r \in R$. Note that if M is a content R-module and $x \in M$, then $x \in c(x)M$ and if $r \in R$, we have $rx \in rc(x)M$ and therefore $c(rx) \subseteq rc(x)$. The question arises when the equality holds. The answer to this question will be given in Theorem 1.1.4.

For $f, g \in R[X]$, it is easy to check that $c(fg) \subseteq c(f)c(g)$ and it is of interest to know when the equality holds. For $f, g \in R[X]$, the Dedekind-Mertens Lemma states that $c(fg)c(g)^m = c(f)c(g)c(g)^m$ for some $m \geq 0$ depending on f [**AG**]. A polynomial f is said to be Gaussian if $c(fg) = c(f)c(g)$ for all $g \in R[X]$. For example if the $c(f)$ is a cancellation ideal of R, then f is Gaussian (An ideal I of R is said to be cancellation if $IJ = IK$ causes $J = K$ for all ideals J and K of R). Also note that we may not have the equality as the following example shows: Let $R = \mathbb{Z} + 2i\mathbb{Z}$ and $f = g = 2i + 2X$. Then it is easy to check that $c(fg) = (4)$, while $c(f)c(g) = (4, 4i)$. However $c(fg)c(g) = c(f)c(g)c(g) = (8, 8i)$.

"In [**Ed**, page 2], Edwards states that the theorem which Dedekind proved (See [**De**]) said that if f and g are polynomials whose coefficients are algebraic numbers such that the coefficients of fg are algebraic integers, then the product of any coefficient of f with an arbitrary coefficient of g is also an algebraic integer. In modern language this is just saying that the ideal $c(f)c(g)$ is integral over the ideal $c(fg)$, a theorem which is considerably weaker than what is now known as the Dedekind-Mertens Lemma. Apparently the fact that there exists an N such that $c(f)^N c(f)c(g) = c(f)^N c(fg)$ was known to Mertens, Kronecker [**Kro**], and perhaps Hurwitz [**Hur**]. Mertens' proof in 1892 shows that in characteristic 0, the number N can be chosen approximately equal to $\deg(g)^2$. However, the precise value $N = \deg(g)$ was given by Dedekind in his 1892 paper [**De**, page 10]. Prüfer reproved this theorem with $N = \deg(g)$ [**Pr**, page 24] in 1932. The earliest reference we have found with the name 'Dedekind-Mertens Lemma' is in Krull [**Kru**, page 128]. It is interesting that the theorem Krull states as the Dedekind-Mertens Lemma has the value N equal to the sum of the degrees of f and g. In an article in 1959 [**No**] Northcott says he has not been able to find a reference for the Dedekind-Mertens Lemma and gives a proof he attributes to Emil Artin. Arnold and Gilmer [**AG**] give a proof and a generalization whose idea we use in this paper." [**HH**, p. 1306]

Gilmer, Grams and Parker [**GGP**, Theorem 3.6, 3.7] and Anderson and Kang [**AK**] proved and generalized the Dedekind-Mertens Lemma: if g has $k+1$ nonzero terms, then $c(f)^k c(f)c(g) = c(f)^k c(fg)$ holds for all polynomials $f \in R[X]$. Heinzer and Huneke define an interesting notion called the Dedekind-Mertens number of $\mu(g)$ of a polynomial $g \in R[X]$ that is the smallest positive integer k such that $c(fg)c(g)^{k-1} = c(f)c(g)c(g)^{k-1}$ and sharpen the upper bound for $\mu(g)$ replacing the number of nonzero coefficients by the maximum of the number of the minimal generators of $c(g)R_{\mathbf{m}}$, where $\mathbf{m}$ runs over all maximal ideals of R [**HH**, Theorem 2.1].

There are many works on the Dedekind-Mertens Lemma. One may refer to [**AG**], [**AK**], [**BG**], [**GGP**], [**HH**], [**No**] and [**T**]. However among all these works, a nice generalization of the Dedekind-Mertens Lemma in [**No**] is of our special interest, since it is a good example for a concept related to the Dedekind-Mertens Lemma, the notion of the so-called *content algebra*.

Let S be a commutative semigroup. The canonical form of an element f in the semigroup ring $R[S]$ is of the form $f = a_0X^{s_0} + a_1X^{s_1} + \cdots + a_nX^{s_n}$, where $a_i \in R$ and $s_i \in S$ for all $0 \leqslant i \leqslant n$. The content of f is the ideal of R generated by the coefficients of f, i.e. $c(f) = (a_0, a_1, \ldots, a_n)$.

Northcott in [**No**] proves that if S is a commutative, cancellative and torsion-free semigroup, then for all $f, g \in R[S]$, there exists an $m \geqslant 0$ such that

$$c(fg)c(g)^m = c(f)c(g)c(g)^m.$$

By the Dedekind-Mertens Lemma, one may give a very simple proof for McCoy's Theorem for the zero-divisors of semigroup rings. McCoy's Theorem says that if f is a zero-divisor of $R[S]$, then there exists a nonzero $r \in R$ such that $f.r = 0$ or equivalently $c(f).r = (0)$. The proof is as follows: Assume that $f \in R[S]$ and $g \in R[S] - \{0\}$, such that $fg = 0$. So there exists a natural number k such that $c(f)^k c(g) = c(f)^{k-1} c(fg) = (0)$. Take t the smallest natural number such that $c(f)^t c(g) = (0)$ and choose r a nonzero element of $c(f)^{t-1}c(g)$. It is easy to check that $c(f).r = (0)$ and equivalently $f.r = 0$.

This short introduction and some other examples and results in the thesis will show us that it is useful to consider content algebras:

Let R be a commutative ring with identity and R' an R-algebra. R' is defined to be a *content R-algebra*, if the following conditions hold:

(1) R' is a content R-module.
(2) (*Faithful flatness*) For any $r \in R$ and $f \in R'$, the equation $c(rf) = rc(f)$ holds and $c(R') = R$.
(3) (*Dedekind-Mertens content formula*) For each f and g in R', there exists a natural number n such that $c(f)^n c(g) = c(f)^{n-1} c(fg)$.

Chapter 2 begins by the definition of content and weak content algebras and we prove that if R is a ring and S, a commutative monoid, then the monoid ring $B = R[S]$ is a content R-algebra if and only if one of the following conditions satisfies:

(1) For $f, g \in B$, if $c(f) = c(g) = R$, then $c(fg) = R$.
(2) (*McCoy's Property*) For $g \in B$, g is a zero-divisor of B iff there exists $r \in R - \{0\}$ such that $rg = 0$.
(3) S is a cancellative and torsion-free monoid.

In 2.2, we discuss prime ideals of content and weak content algebras and we show that in content extensions, minimal primes extend to minimal primes. More precisely, if B is a content R-algebra, then there is a correspondence between $\mathrm{Min}(R)$ and $\mathrm{Min}(B)$, with the function $\varphi : \mathrm{Min}(R) \longrightarrow \mathrm{Min}(B)$ defined by $\mathbf{p} \longrightarrow \mathbf{p}B$.

In 2.3, we introduce a family of rings which have very few zero-divisors. It is a well-known result that the set of zero-divisors of a Noetherian ring is a finite union of its associated primes [**K**, p. 55]. Rings having few zero-divisors have been introduced in [**Dav**]. We define that a ring R has *very few zero-divisors*, if $Z(R)$ is a finite union of prime ideals in $\mathrm{Ass}(R)$. In this section, we prove that if B is a content R-algebra, then R has very few zero-divisors iff B has very few zero-divisors.

Another celebrated property of Noetherian rings is that every ideal entirely contained in the set of their zero-divisors has a nonzero annihilator. A ring R has *Property (A)*, if each finitely generated ideal $I \subseteq Z(R)$ has a nonzero annihilator [**HK**]. In Section 2.3, we also prove some results for content algebras over rings having Property (A) and then we discuss rings having few zero-divisors in more details. Let us recall that a ring R is said to have few zero-divisors, if the set $Z(R)$ of zero-divisors is a finite union of prime ideals. It is well-known that a ring R has few zero-divisors iff its classical quotient ring $T(R)$ is semi-local [**Dav**]. We may suppose that $Z(R) = \bigcup_{i=1}^{n} \mathbf{p}_i$ such that $\mathbf{p}_i \nsubseteq \bigcup\cup_{j=1 \wedge j \neq i}^{n} \mathbf{p}_j$ for all $1 \leq i \leq n$. Then we have $\mathbf{p}_i \nsubseteq \mathbf{p}_j$ for all $i \neq j$ and by the Prime Avoidance Theorem, these prime ideals are uniquely determined. In such a case, it is easy to see that $\mathrm{Max}(T(R)) = \{\mathbf{p}_1 T(R), \ldots, \mathbf{p}_n T(R)\}$, where by $T(R)$ we mean total quotient ring of R. Such prime ideals are called maximal primes in $Z(R)$. We denote the number of maximal primes in $Z(R)$ by $\mathrm{zd}(R)$. As one of the main results of this section, we show that if R has Property (A) and $\mathrm{zd}(R) = n$ and B is a content R-algebra, then $\mathrm{zd}(B) = n$. At the end of this section, we consider the interesting case, when $\mathrm{zd}(R) = 1$, i.e. $Z(R)$ is an ideal of R. Such a ring is called a primal ring [**Dau**].

We let $Z(R)^*$ denote the (nonempty) set of proper zero-divisors of R, where by a proper zero-divisor we mean a zero-divisor different from zero. We consider the graph $\Gamma(R)$, called the zero-divisor graph of R, whose vertices are the elements of $Z(R)^*$ and edges are those pairs of distinct proper zero-divisors $\{a, b\}$ such that $ab = 0$. Section 2.4 is devoted to examine the preservation of diameter of zero-divisor graph under content extensions.

In 2.5, we discuss Gaussian and Armendariz content algebras that are natural generalization of the same concepts in polynomials rings. In this section, we show that if B is a content R-algebra, then B is a Gaussian R-algebra iff for any ideal I of R, B/IB is an Armendariz (R/I)-algebra. This is a generalization of a result in [**AC**].

In 2.6, we prove some results about a special content algebra, i.e. $l(R, B)$, a generalization of the algebra $R(X) = R[X]_S$, where $S = \{f \in R[X] : c(f) = R\}$.

Some of the results for zero-divisors of content algebras can be proved for semigroup modules. Section 2.7 is devoted to zero-divisors of semigroup modules. Let S be a commutative semigroup and M be an R-module. One can define the semigroup module $M[S]$ as an $R[S]$-module constructed from the semigroup S and the R-module M simply similar to standard definition of semigroup rings. Obviously similar to semigroup rings, the zero-divisors of the semigroup module $M[S]$ are interesting to investigate ([**G1**, p. 82] and [**J**]).

Chapter 3 is devoted to the grade of zero-divisor modules. For doing that we need to know about *locally Nakayama module* and investigate modules having very few zero-divisors with another approach.

In 1.2, locally Nakayama modules have been introduced and we have obtained a necessary and sufficient condition for $\mathrm{Hom}_{R_p}(N_p, M_p) \neq 0$. An R-module M is said to be *locally Nakayama* if $M_p \neq 0$ implies that $M_p/pM_p \neq 0$, for all $p \in \mathrm{Spec}(R)$.

In 3.1, we discuss modules having very few zero-divisors with another view and construct enough materials to generalize some interesting results about the grade of a module. Let I be a finitely generated ideal of R. We can define $\mathrm{grade}(I, M)$ in two different ways: either by the notion of the longest M-sequence in I, when R is a Noetherian ring and M a finitely generated R-module or by the homological grade which is defined by using the Koszul complex. In [**BH**, Theorem 1.6.17], it has been shown that for a finitely generated module over a Noetherian ring the second definition of grade is consistent with the first one. The question arises whether these two definitions are consistent for a larger class of modules as well.

In section 3.2, we give a large class of modules, containing finitely generated modules, for which these two definitions coincide. An R-module M has *very few zero-divisors*, if $Z_R(M)$, the set of zero divisors of M over R, is a finite union of prime ideals in $\mathrm{Ass}_R(M)$, where by $\mathrm{Ass}_R(M)$ we mean the set of associated prime ideals of the R-module M. Let I be a finitely generated ideal of R and M an R-module such that $M \neq IM$. We show that if $M/\mathbf{x}M$ has very few zero-divisors for any ideal $\mathbf{x}$ of R generated by an M-sequence in I, then the length of a maximal M-sequence in I is equal to $\mathrm{grade}(I, M)$ defined by the concept of Koszul complex. Also, we prove that if $M/\mathbf{x}M$ has very few zero-divisors for any ideal $\mathbf{x}$ of R generated by an M-sequence in I, then $\mathrm{grade}(I, M)$ defined by the concept of Koszul complex is the least integer i such that $H_I^i(M) \neq 0$, where $H_I^i(M) = \varinjlim_{n \in \mathbb{N}} \mathrm{Ext}_R^i(R/I^n, M)$ is i-th local cohomology module of M with respect to I. The reader can refer to [**BS**], for the basic properties of local cohomology modules. Note that this is a generalization of a well-known theorem in theory of local cohomology of finitely generated modules over Noetherian rings [**BS**, Theorem 6.2.7].

An R-module M is said to be a *zero-divisor module* (ZD-module), if M/N has very few zero-divisors, for any submodule N of M. According to [**DE**, Example 2.2], the class of ZD-modules is much larger than that of finitely generated modules. As one of the interesting results of this section, we prove that if R is Noetherian, and M a ZD-module, then for all submodules N of M, the following statement are equivalent:

(1) $\mathrm{Ass}_R(H_I^0(M/N)/L)$ is finite for all finitely generated submodules L of $H_I^0(M/N)$;

(2) if $H_I^0(M/N), \ldots, H_I^{i-1}(M/N)$ are finitely generated, then $\mathrm{Ass}_R(H_I^i(M/N)/L)$ is finite for all finitely generated submodules L of $H_I^i(M/N)$.

Some parts of the section 3.1 and the sections 3.2 and 1.2 come from the joint paper of the author of this thesis and Payrovi with the title "Modules having very few zero-divisors", published at Communications in Algebra, Volume 38, Issue 9, Pages 3154–3162 in 2010. The main content of the sections 2.1, 2.2, 2.3 and 2.4 and the whole section 2.7 come from the paper "Zero-divisors of content algebras" published at Archivum Mathematicum (Brno), Volume 46, Issue 4, Pages 237–249 in 2010 and the paper "Zero-divisors of semigroup modules" to appear in Kyungpook Mathematical Journal, respectively, both composed by the author of this thesis.

Unless otherwise stated, our notation and terminology will follow as closely as possible that of Gilmer [**G1**].

CHAPTER 1

Preliminaries

1.1. Content modules

In this section, first we give the definition of content modules with some results that we need in the rest of the thesis. More on content modules can be found in [**OR**] and [**ES**].

DEFINITION 1.1.1. Let R be a commutative ring with identity, and M a unitary R-module and the *content function*, c from M to the ideals of R defined by

$$c(x) = \bigcap \{I \colon I \text{ is an ideal of } R \text{ and } x \in IM\}.$$

M is called a *content R-module* if $x \in c(x)M$, for all $x \in M$, also when N is a non-empty subset of M, then by $c(N)$ we mean the ideal generated by all $c(x)$ that $x \in N$ ([**OR**, Definition 1.1]).

LEMMA 1.1.2. *Let M be an R-module. The following statements are equivalent:*

(1) *M is a content R-module, i.e. $x \in c(x)M$, for all $x \in M$.*
(2) *For any non-empty family of ideals $\{I_i\}$ of R, $(\bigcap I_i)M = \bigcap (I_i M)$.*

Moreover when M is a content R-module, $c(x)$ is a finitely generated ideal of R, for all $x \in M$ ([**OR**, Proposition 1.2]*).*

PROOF. $(1) \to (2)$: Let $x \in \bigcap (I_i M)$. Therefore $x \in I_i M$ for every i. This implies that $c(x) \subseteq I_i$ for every i, which causes $c(x) \subseteq \bigcap I_i$ and since M is a content R-module, we have $x \in (\bigcap I_i)M$.

$(2) \to (1)$ is obvious.

Now let $x \in M$, therefore $x = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$, where $c_i \in c(x)$ and $x_i \in M$ for all $0 \leq i \leq n$. So $x \in (c_1, c_2, \ldots, c_n)M$ and obviously $c(x) \subseteq (c_1, c_2, \ldots, c_n)$. But $(c_1, c_2, \ldots, c_n) \subseteq c(x)$. □

THEOREM 1.1.3. ***Nakayama Lemma for Content Modules**: Let M be a content R-module and* $\operatorname{Jac}(R)$ *be the Jacobson radical of R and I be an ideal of R such that $I \subseteq$* $\operatorname{Jac}(R)$. *If $IM = M$, then $M = (0)$.*

PROOF. Let $x \in M$. Since M is a content R-module, $x \in c(x)M$, but $IM = M$, so $x \in c(x)IM$ and therefore $c(x) \subseteq c(x)I$, but $c(x)$ is a finitely generated ideal of R, so by Nakayama lemma for finitely generated modules, $c(x) = (0)$ and consequently $x = 0$. $\square$

From the definition of the content of a polynomial, it is obvious that $c(rf) = rc(f)$ for all $f \in R[X]$ and $r \in R$. Note that if M is a content R-module and $x \in M$, then $x \in c(x)M$ and if $r \in R$, we have $rx \in rc(x)M$ and therefore $c(rx) \subseteq rc(x)$. The question arises when the equality for all r and x holds.

THEOREM 1.1.4. *Let M be a content R-module. Then the following are equivalent:*

(1) *For every $r \in R$ and $x \in M$, $rc(x) = c(rx)$.*
(2) *$(I : r)_R M = (IM : r)_M$ for every ideal I of R and $r \in R$.*
(3) *$(0 : r)_R M = (0M : r)_M$ for every $r \in R$*
(4) *M is flat.*

Moreover, M is faithfully flat iff M is flat and $c(M) = R$ ([**OR**, Theorem 1.5 and Corollary 1.6]*).*

PROOF. $(1) \to (2)$: The inclusion $\subseteq$ is always true. Now let $x \in IM : r$. Therefore $rx \in IM$, and hence $c(rx) \subseteq I$. But $rc(x) = c(rx)$ by (1), so $c(x) \subseteq I : r$ and $x \in (I : r)M$, since M is a content R-module.

$(2) \to (3)$ is obvious.

$(3) \to (1)$: Clearly $rc(x) \supseteq c(rx)$. Also $rx \in rM$ implies $c(rx) \subseteq (r)$. Therefore $c(rx) = (r)B$, where $B = c(rx) : r$. But $rx \in c(rx)M = rBM$. Therefore $rx = rz$ for some element $z \in BM$, which implies $x - z \in 0M : r$. But $(0 : r)_R M = (0M : r)_M$, so then $x - z \in (0 : r)M \subseteq BM$ and therefore $x \in BM$. This implies $c(x) \subseteq B$, and hence $rc(x) \subseteq rB = c(rx)$.

$(4) \leftrightarrow (1)$: This is because M is flat iff $(I : r)_R M = (IM : r)_M$ for every ideal I of R and $r \in R$ [**B**, p. 65. Ex. 22].

For the second assertion suppose M is flat and $c(M) = R$, then obviously $\mathbf{m}M \neq M$ for every maximal ideal of R (Note that it is easy to prove that when M is a content and flat R-module, then $c(IM) = Ic(M)$ for every ideal I of R). Therefore M is faithfully flat by [**B**, p. 44, Props. 1]. Conversely if M is faithfully flat, then M is flat and $\mathbf{m}M \neq M$ for every maximal ideal of R. Therefore for every maximal ideal $\mathbf{m}$ of R, there exists an $x \in M$ such that $x \notin \mathbf{m}M$. Therefore $c(M) \subsetneq \mathbf{m}$ for every maximal ideal of R, and consequently $c(M) = R$. $\square$

Let M be a content R-module and define $S_M = \{r \in R\colon c(rx) = rc(x) \text{ for all } x \in M\}$. It is obvious that S_M is multiplicatively closed. Now we give the following important theorem:

THEOREM 1.1.5. *Let M be a content R-module and consider a multiplicatively closed set $S \subseteq S_M$ of R. Then M_S is a content R_S-module with $c_M(x)R_S = c_{M_S}(x/s)$. Particularly when L is a flat content R-module and T is a multiplicatively closed set of R, then L_T is a flat content R_T-module as well (*[**OR**, Theorem 3.1]*).*

With the help of the above theorems, one may describe some of the prime and primary submodules of faithfully flat and content modules.

DEFINITION 1.1.6. Let M be an R-module and P be a proper R-submodule of M. P is said to be a *prime submodule* of M, if $rx \in P$ implies $x \in P$ or $rM \subseteq P$, for each $r \in R$ and $x \in M$.

DEFINITION 1.1.7. Let M be an R-module and P be a proper R-submodule of M. P is said to be a *primary submodule* of M, if $rx \in P$ then $x \in P$ or there exists a natural number n such that $r^n M \subseteq P$, for each $r \in R$ and $x \in M$.

THEOREM 1.1.8. *Let M be a content and faithfully flat R-module and $\mathbf{p}$ be an ideal of R. Then $\mathbf{p}M$ is a primary (prime) R-submodule of M iff $\mathbf{p}$ is a primary (prime) ideal of R.*

PROOF. Let $\mathbf{p}$ be a prime ideal of R and $r \in R$ and $x \in M$ such that $rx \in \mathbf{p}M$. Therefore $c(rx) \subseteq \mathbf{p}$ and since $c(rx) = rc(x)$ we have $rc(x) \subseteq \mathbf{p}$ and this means that $c(x) \subseteq \mathbf{p}$ or $(r) \subseteq \mathbf{p}$ and at last $x \in \mathbf{p}M$ or $rM \subseteq \mathbf{p}M$. Notice that since M is a faithfully flat R-module, $\mathbf{p}M \neq M$. The other assertions can be proved in a similar way. □

1.2. Locally Nakayama modules

Recall that Nakayama's lemma plays an important role in commutative algebra and one of the most useful applications of this lemma is:

THEOREM 1.2.1. *If R is a Noetherian ring and M, N are finitely generated R-modules, then $\mathrm{Hom}_R(N,M) = 0$ implies that $\mathrm{Ann}_R(N) \not\subseteq Z_R(M)$ (*[**BH**, Proposition 1.2.3]*).*

We define locally Nakayama modules and later in Theorem 3.1.5, we obtain a necessary and sufficient condition for $\mathrm{Hom}_{R_\mathbf{p}}(N_\mathbf{p}, M_\mathbf{p}) \neq 0$.

DEFINITION 1.2.2. An R-module M is said to be *locally Nakayama* if $M_\mathbf{p} \neq 0$ implies that $M_\mathbf{p}/\mathbf{p}M_\mathbf{p} \neq 0$, for all $\mathbf{p} \in \mathrm{Spec}(R)$.

REMARK 1.2.3. Let M be an R-module. The condition that $M_\mathbf{p}/\mathbf{p}M_\mathbf{p} \neq 0$ is caused by $M_\mathbf{p} \neq 0$, for all $\mathbf{p} \in \mathrm{Max}(R)$ is not sufficient for the R-module M to be locally Nakayama as the following example - suggested by Holger Brenner - shows us:

Let $R = k[[X,Y]]$, where k is a field and define a ring R-homomorphism $\varphi : R \longrightarrow R$ by $X \longmapsto X$ and $Y \longmapsto XY$, which gives us an R-algebra A. Also put $\mathbf{p} = (X,Y)$ and

$\mathbf{q} = (X)$. It is easy to check that $\mathbf{p}$ is the only maximal ideal of R and $\mathbf{q}$ a prime ideal with $A_\mathbf{p}/\mathbf{p}A_\mathbf{p} \neq 0$, while $A_\mathbf{q}/\mathbf{q}A_\mathbf{q} = 0$.

THEOREM 1.2.4. *Let R be a ring. The following statements hold:*

(1) *Every finitely generated R-module is locally Nakayama.*
(2) *If C is a content and flat R-module, then it is locally Nakayama.*
(3) *If M and N are locally Nakayama R-modules, then $M \oplus N$ is a locally Nakayama R-module.*
(4) *If M and N are locally Nakayama R-modules, then $M \otimes_R N$ is also a locally Nakayama R-module and* $\mathrm{Supp}(M \otimes_R N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$.
(5) *Every module over a von Neumann regular ring is locally Nakayama. Note that a ring R is said to be von Neumann regular if for every $a \in R$ there exists an $x \in R$ with $a = a^2 x$.*

PROOF. (1) Let M be a finitely generated R-module and $\mathbf{p} \in \mathrm{Spec}(R)$. Then $M_\mathbf{p}$ is a finitely generated R_p-module. By Nakayama's lemma if $M_\mathbf{p} \neq 0$ then $M_\mathbf{p}/\mathbf{p}M_\mathbf{p} \neq 0$.

(2) If C is a content and flat R-module, then by Theorem 1.1.5 $C_\mathbf{p}$ is a content and flat $R_\mathbf{p}$-module, for all $\mathbf{p} \in \mathrm{Spec}(R)$. Hence by Nakayama's lemma for content modules [Theorem, 1.1.3], if $C_\mathbf{p} \neq 0$ then $C_\mathbf{p}/\mathbf{p}C_\mathbf{p} \neq 0$.

(3) Let $(M \oplus N)_\mathbf{p} \neq 0$. Without losing any generality, we can assume that $M_\mathbf{p} \neq 0$. Consider the following exact sequence:

$$0 \longrightarrow M_\mathbf{p}/\mathbf{p}M_\mathbf{p} \longrightarrow (M \oplus N)_\mathbf{p}/\mathbf{p}(M \oplus N)_\mathbf{p}.$$

Since M is a locally Nakayama R-module, $M \oplus N$ is a locally Nakayama R-module.

(4) Let M and N be locally Nakayama R-modules and $(M \otimes N)_\mathbf{p} \neq 0$. Then $M_\mathbf{p} \otimes N_\mathbf{p} \neq 0$ and therefore $M_\mathbf{p} \neq 0$ and $N_\mathbf{p} \neq 0$. By assumption $M_\mathbf{p}/\mathbf{p}M_\mathbf{p} \neq 0$, $N_\mathbf{p}/\mathbf{p}N_\mathbf{p} \neq 0$ and both are $k(\mathbf{p})$-vector spaces and consequently $M_\mathbf{p}/\mathbf{p}M_\mathbf{p} \otimes_{k(\mathbf{p})} N_\mathbf{p}/\mathbf{p}N_\mathbf{p} \neq 0$.

Now we have:

$$\begin{aligned} M_\mathbf{p}/\mathbf{p}M_\mathbf{p} \otimes_{k(\mathbf{p})} N_\mathbf{p}/\mathbf{p}N_\mathbf{p} &\cong (M/\mathbf{p}M \otimes_{R/\mathbf{p}} N/\mathbf{p}N)_\mathbf{p} \\ &\cong (M \otimes_R R/\mathbf{p} \otimes_R N)_\mathbf{p} \\ &\cong (M \otimes N)_\mathbf{p}/\mathbf{p}(M \otimes N)_\mathbf{p}. \end{aligned}$$

So $(M \otimes N)_\mathbf{p}/\mathbf{p}(M \otimes N)_\mathbf{p} \neq 0$ and $M \otimes N$ is locally Nakayama.

For the second part, let $\mathbf{p} \in \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$. But M and N are locally Nakayama, so $M_\mathbf{p}/\mathbf{p}M_\mathbf{p} \otimes_{k(\mathbf{p})} N_\mathbf{p}/\mathbf{p}N_\mathbf{p} \neq 0$. Therefore by the above isomorphisms $(M \otimes N)_\mathbf{p} \neq 0$. This means that $\mathrm{Supp}(M) \cap \mathrm{Supp}(N) \subseteq \mathrm{Supp}(M \otimes N)$. The other inclusion is obvious.

(5) Let R be a von Neumann regular ring and M a nonzero R-module. Assume $\mathbf{p}$ is a prime ideal of R such that $M_\mathbf{p} \neq 0$. So there exists a cyclic submodule N of M such that $N_\mathbf{p} \neq 0$, and by Nakayama's lemma $N_\mathbf{p}/\mathbf{p}N_\mathbf{p} \neq 0$. The exact sequence $0 \longrightarrow$

$N \longrightarrow M$ induces the exact sequence $0 \longrightarrow N_{\mathbf{p}}/\mathbf{p}N_{\mathbf{p}} \longrightarrow M_{\mathbf{p}}/\mathbf{p}M_{\mathbf{p}}$, by [**Ro**, Theorem 4.16]. Therefore $M_{\mathbf{p}}/\mathbf{p}M_{\mathbf{p}} \neq 0$ and consequently M is locally Nakayama. $\square$

CHAPTER 2

Content algebras

2.1. Content and weak content algebras

Content algebras and later weak content algebras were introduced and discussed in [**OR**] and [**Ru**] respectively. Content algebras are actually a natural generalization of polynomial rings as we discussed in the introduction of the thesis. Let us recall that if R is a ring, then for each f and g in $R[X]$, there exists a natural number n such that $c(f)^n c(g) = c(f)^{n-1} c(fg)$ [**AG**]. Now we bring the definition of content algebras from [**OR**, 6, p. 63]:

DEFINITION 2.1.1. Let R be a commutative ring with identity and R' an R-algebra. R' is defined to be a *content R-algebra*, if the following conditions hold:

(1) R' is a content R-module.
(2) (*Faithful flatness*) For any $r \in R$ and $f \in R'$, the equation $c(rf) = rc(f)$ holds and $c(R') = R$.
(3) (*Dedekind-Mertens content formula*) For each f and g in R', there exists a natural number n such that $c(f)^n c(g) = c(f)^{n-1} c(fg)$.

A good example of a content R-algebra is the monoid ring $R[M]$ where M is a commutative, cancellative and torsion-free monoid [**No**]. This is actually a free R-algebra. For some examples of content R-algebras that as R-modules are not free, refer to [**OR**, Examples 6.3, p. 64]. Rush defined weak content algebras as follows [**Ru**, p. 330]:

DEFINITION 2.1.2. Let R be a commutative ring with identity and R' an R-algebra. R' is defined to be a *weak content R-algebra*, if the following conditions hold:

(1) R' is a content R-module.
(2) (*Weak content formula*) For all f and g in R', $c(f)c(g) \subseteq \operatorname{rad}(c(fg))$ (Here $\operatorname{rad}(A)$ denotes the radical of the ideal A).

And he gave an equivalent condition for when an algebra that is a content module is a weak content algebra [**Ru**, Theorem 1.2, p. 330]:

THEOREM 2.1.3. *Let R' be an R-algebra such that R' is a content R-module. The following are equivalent:*

(1) *R' is a weak content R-algebra.*

(2) *For each prime ideal* $\mathbf{p}$ *of* R*, either* $\mathbf{p}R'$ *is a prime ideal of* R'*, or* $\mathbf{p}R' = R'$.

It is obvious that content algebras are weak content algebras, but the converse is not true. For example if R is a Noetherian ring, then $R[[X_1, X_2, \ldots, X_n]]$ is a weak content R-algebra, while it is not necessarily a content R-algebra as the following example taken from [**Ru**, p. 331] shows: Let $R = k[u,v]$, k a field and u, v indeterminates. Let $f = u + vX + vX^2 + \cdots$, and $g = v + X \in R[[X]]$. Then $c(g) = R$ but $c(fg) = (uv, u + v^2, v + v^2) \neq (u,v) = c(f)$. Thus $c(g)^{n+1}c(f) \neq c(g)^n c(fg)$ for each $n \geq 1$. We end our brief introduction to content algebras with the following theorem:

THEOREM 2.1.4. *Let* R *be a ring and* S *be a commutative monoid. Then the following statements about the monoid algebra* $B = R[S]$ *are equivalent:*

(1) *B is a content R-algebra.*
(2) *B is a weak content R-algebra.*
(3) *For $f, g \in B$, if $c(f) = c(g) = R$, then $c(fg) = R$.*
(4) *(McCoy's Property) For $g \in B$, g is a zero-divisor of B iff there exists $r \in R - \{0\}$ such that $rg = 0$.*
(5) *S is a cancellative and torsion-free monoid.*

PROOF. $(1) \to (2) \to (3)$ and $(1) \to (4)$ are obvious ([**OR**] and [**Ru**]). Also, according to [**No**] (5) implies (1). Therefore the proof will be complete if we prove that (3) as well as (4) implies (5).

$(3) \to (5)$: We prove that if S is not cancellative or not torsion-free then (3) cannot hold. For the moment, suppose that S is not cancellative, so there exist $s, t, u \in S$ such that $s + t = s + u$ while $t \neq u$. Put $f = X^s$ and $g = (X^t - X^u)$. Then obviously $c(f) = c(g) = R$, while $c(fg) = (0)$. Finally suppose that S is cancellative but not torsion-free. Let $s, t \in S$ be such that $s \neq t$, while $ns = nt$ for some natural n. Choose the natural number k minimal so that $ks = kt$. Then we have $0 = X^{ks} - X^{kt} = (X^s - X^t)(\sum_{i=0}^{k-1} X^{(k-i-1)s+it})$.

Since S is cancellative, the choice of k implies that $(k - i_1 - 1)s + i_1 t \neq (k - i_2 - 1)s + i_2 t$ for $0 \leq i_1 < i_2 \leq k-1$. Therefore $\sum_{i=0}^{k-1} X^{(k-i-1)s+it} \neq 0$, and this completes the proof. In a similar way one can prove $(4) \to (5)$ [**G2**, p.82]. □

2.2. Prime ideals in content algebras

Let B be a weak content R-algebra such that for all $\mathbf{m} \in \mathrm{Max}(R)$ (by $\mathrm{Max}(R)$, we mean the maximal ideals of R), we have $\mathbf{m}B \neq B$, then by Theorem 2.1.3, prime ideals extend to prime ideals. Particularly in content algebras primes extend to primes. We recall that when B is a content R-algebra, then g is a zero-divisor of B, iff there exists an $r \in R - \{0\}$ such that $rg = 0$ [**OR**, 6.1, p. 63]. We give the following theorem about associated prime ideals. We assert that by $\mathrm{Ass}_R(M)$, we mean the associated prime ideals of the R-module M.

THEOREM 2.2.1. *Let B be a content R-algebra and M a nonzero R-module. If $\boldsymbol{p} \in \mathrm{Ass}_R(M)$ then $\boldsymbol{p}B \in \mathrm{Ass}_B(M \otimes_R B)$.*

PROOF. Let $\mathbf{p} \in \mathrm{Ass}_R(M)$ and $\mathbf{p} = \mathrm{Ann}(x)$, where $x \in M$. Therefore $0 \longrightarrow R/\mathbf{p} \longrightarrow M$ is an R-exact sequence. Since B is a faithfully flat R-module, we have the following B-exact sequence:

$$0 \longrightarrow B/\mathbf{p}B \longrightarrow M \otimes_R B$$

with $\mathbf{p}B = \mathrm{Ann}(x \otimes_R 1_B)$. Since B is a content R-algebra, $\mathbf{p}B$ is a prime ideal of B. □

Now we give a general theorem on minimal prime ideals in algebras. One of the results of this theorem is that in faithfully flat weak content algebras (including content algebras), minimal primes extend to minimal primes and, more precisely, there is actually a correspondence between the minimal primes of the ring and their extensions in the algebra.

THEOREM 2.2.2. *Let B be an R-algebra with the following properties:*

(1) *For each prime ideal $\boldsymbol{p}$ of R, the extended ideal $\boldsymbol{p}B$ of B is prime.*
(2) *For each prime ideal $\boldsymbol{p}$ of R, $\boldsymbol{p}B \cap R = \boldsymbol{p}$.*

Then the function $\varphi : \mathrm{Min}(R) \longrightarrow \mathrm{Min}(B)$ given by $\boldsymbol{p} \longrightarrow \boldsymbol{p}B$ is a bijection.

PROOF. First we prove that if $\mathbf{p}$ is a minimal prime ideal of R, then $\mathbf{p}B$ is also a minimal prime ideal of B. Let Q be a prime ideal of B such that $Q \subseteq \mathbf{p}B$. So $Q \cap R \subseteq \mathbf{p}B \cap R = \mathbf{p}$. Since $\mathbf{p}$ is a minimal prime ideal of R, we have $Q \cap R = \mathbf{p}$ and therefore $Q = \mathbf{p}B$. This means that φ is a well-defined function. Obviously the second condition causes φ to be one-to-one. The next step is to prove that φ is onto. For showing this, consider $Q \in \mathrm{Min}(B)$, so $Q \cap R$ is a prime ideal of R such that $(Q \cap R)B \subseteq Q$ and therefore $(Q \cap R)B = Q$. Our claim is that $(Q \cap R)$ is a minimal prime ideal of R. Suppose $\mathbf{p}$ is a prime ideal of R such that $\mathbf{p} \subseteq Q \cap R$, then $\mathbf{p}B \subseteq Q$ and since Q is a minimal prime ideal of B, $\mathbf{p}B = Q = (Q \cap R)B$ and therefore $\mathbf{p} = Q \cap R$. □

COROLLARY 2.2.3. *Let B be a weak content and faithfully flat R-algebra. Then the function $\varphi : \mathrm{Min}(R) \longrightarrow \mathrm{Min}(B)$ given by $\boldsymbol{p} \longrightarrow \boldsymbol{p}B$ is a bijection.*

PROOF. Since B is a weak content and faithfully flat R-algebra, then for each prime ideal $\mathbf{p}$ of R, the extended ideal $\mathbf{p}B$ of B is prime and also $c(1_B) = R$ by [**OR**, Corollary 1.6] and Theorem 2.0.7. Now consider $r \in R$, then $c(r) = c(r.1_B) = r.c(1_B) = (r)$. Therefore if $r \in IB \cap R$, then $(r) = c(r) \subseteq I$. Thus for each prime ideal $\mathbf{p}$ of R, $\mathbf{p}B \cap R = \mathbf{p}$. □

COROLLARY 2.2.4. *Let R be a Noetherian ring. Then the function $\varphi : \mathrm{Min}(R) \longrightarrow \mathrm{Min}(R[[X_1,\ldots,X_n]])$ given by $\boldsymbol{p} \longrightarrow \boldsymbol{p}.(R[[X_1,\ldots,X_n]])$ is a bijection.*

We end this section with a theorem on the height of prime ideals.

THEOREM 2.2.5. *Let B be a ring extension of R such that the following conditions hold:*

(1) *For each prime ideal $\boldsymbol{p}$ of R, the extended ideal $\boldsymbol{p}B$ of B is prime.*
(2) *For each prime ideal $\boldsymbol{p}$ of R, $\boldsymbol{p}B \cap R = \boldsymbol{p}$.*
(3) *The Krull dimension of the ring B is finite.*

Then the following statements are equivalent:

(1) *For any chain of three prime ideals $Q_1 \supset Q_2 \supset Q_3$ of B we have $Q_1 \cap R \supset Q_3 \cap R$.*
(2) *For any prime ideal Q of B with $\boldsymbol{p} = Q \cap R$, if $Q \supset \boldsymbol{p}B$, then $\mathrm{h}(Q) = \mathrm{h}(\boldsymbol{p}B) + 1$.*

PROOF. $(1) \to (2)$: We use induction on $\mathrm{h}(\mathbf{p})$ to prove the result. First suppose that $\mathrm{h}(\mathbf{p}) = 0$. It follows that $\mathrm{h}(\mathbf{p}B) = 0$, for, if not, we have $Q \supset \mathbf{p}B \supset Q_1$ for some prime ideal Q_1 of B and according to the assumption $Q \cap R \supset Q_1 \cap R$ and this contradicts that $\mathrm{h}(\mathbf{p}) = 0$. Now we prove that $\mathrm{h}(Q) = 1$. In fact for any prime ideal $Q_1 \subset Q$, we must have $Q_1 \cap R = \mathbf{p}$, since $\mathrm{h}(\mathbf{p}) = 0$. Thus, according to (1), a chain of distinct prime ideals $Q \supset Q_1 \supset Q_2$ is impossible and in this case, $\mathrm{h}(Q) = \mathrm{h}(\mathbf{p}B) + 1$, as desired.

Now, let $\mathrm{h}(\mathbf{p}) = m$ for $m > 0$, and assume that the equality $\mathrm{h}(Q) = \mathrm{h}(\mathbf{p}B) + 1$ is true for $k < m$. We claim that, for any prime $Q_1 \subset Q$, $\mathrm{h}(Q_1) \leq \mathrm{h}(\mathbf{p}B)$. In order to prove this claim, let $\mathbf{p}_1 = Q_1 \cap R$. If $\mathbf{p}_1 = \mathbf{p}$, then $\mathbf{p}B \subseteq Q_1 \subset Q$ implies $Q_1 = \mathbf{p}B$ and obviously $\mathrm{h}(Q_1) \leq \mathrm{h}(\mathbf{p}B)$.

On the other hand, if $\mathbf{p}_1 \subset \mathbf{p}$, then either $Q_1 = \mathbf{p}_1 B$ or $Q_1 \supset \mathbf{p}_1 B$. In the first case, of course, $\mathrm{h}(Q_1) = \mathrm{h}(\mathbf{p}_1 B) \leq \mathrm{h}(\mathbf{p}B)$, and in the latter case, we use induction, since we know that $\mathrm{h}(\mathbf{p}_1) < m$ and we have $\mathrm{h}(Q_1) = \mathrm{h}(\mathbf{p}_1 B) + 1 \leq \mathrm{h}(\mathbf{p}B)$.

But from this claim it immediately follows that $\mathrm{h}(Q) \leq \mathrm{h}(\mathbf{p}B) + 1$ and of course, from $Q \supset \mathbf{p}B$, we have $\mathrm{h}(Q) \geq \mathrm{h}(\mathbf{p}B) + 1$ and this completes the proof.

Now, let the Krull dimension of B be finite. We prove that $(2) \to (1)$. In contrary, if there exists a chain of three prime ideals $Q_1 \supset Q_2 \supset Q_3$ of B such that $Q_1 \cap R = Q_3 \cap R = \mathbf{p}$. Then $Q_1 \supset Q_2 \supset Q_3 \supseteq \mathbf{p}B$ and therefore $\mathrm{h}(Q_1) \geq \mathrm{h}(\mathbf{p}B) + 2$. □

COROLLARY 2.2.6. *Let B be a weak content and faithfully flat R-algebra. Then the following statements are equivalent:*

(1) *For any chain of three prime ideals $Q_1 \supset Q_2 \supset Q_3$ of B we have $Q_1 \cap R \supset Q_3 \cap R$.*
(2) *For any prime ideal Q of B with $\boldsymbol{p} = Q \cap R$, if $Q \supset \boldsymbol{p}B$, then $\mathrm{h}(Q) = \mathrm{h}(\boldsymbol{p}B) + 1$.*

2.3. Content algebras over rings having few zero-divisors

For a ring R, by $Z(R)$, we mean the set of zero-divisors of R. In [**Dav**], it has been defined that a ring R has *few zero-divisors*, if $Z(R)$ is a finite union of prime ideals. We present the following definition to prove some other theorems related to content algebras.

DEFINITION 2.3.1. A ring R has *very few zero-divisors*, if $Z(R)$ is a finite union of prime ideals in $\mathrm{Ass}(R)$.

THEOREM 2.3.2. *Let B be a content R-algebra. Then B has very few zero-divisors iff R has very few zero-divisors.*

PROOF. $(\leftarrow)$: Let $Z(R) = \mathbf{p}_1 \cup \mathbf{p}_2 \cup \cdots \cup \mathbf{p}_n$, where $\mathbf{p}_i \in \mathrm{Ass}_R(R)$ for all $1 \leq i \leq n$. We will show that $Z(B) = \mathbf{p}_1 B \cup \mathbf{p}_2 B \cup \cdots \cup \mathbf{p}_n B$. Let $g \in Z(B)$, so there exists an $r \in R - \{0\}$ such that $rg = 0$ and so $rc(g) = (0)$. Therefore $c(g) \subseteq Z(R)$ and according to the Prime Avoidance Theorem, we have $c(g) \subseteq \mathbf{p}_i$, for some $1 \leq i \leq n$ and therefore $g \in \mathbf{p}_i B$. Now let $g \in \mathbf{p}_1 B \cup \mathbf{p}_2 B \cup \cdots \cup \mathbf{p}_n B$ so there exists an i such that $g \in \mathbf{p}_i B$, so $c(g) \subseteq \mathbf{p}_i$ and $c(g)$ has a nonzero annihilator and this means that g is a zero-divisor of B. Note that $\mathbf{p}_i B \in \mathrm{Ass}_B(B)$, for all $1 \leq i \leq n$.

$(\rightarrow)$: Let $Z(B) = \cup_{i=1}^n Q_i$, where $Q_i \in \mathrm{Ass}(B)$ for all $1 \leq i \leq n$. Therefore $Z(R) = \cup_{i=1}^n (Q_i \cap R)$. Without loss of generality, we can assume that $Q_i \cap R \nsubseteq Q_j \cap R$ for all $i \neq j$. Now we prove that $Q_i \cap R \in \mathrm{Ass}(R)$ for all $1 \leq i \leq n$. Consider $f \in B$ such that $Q_i = \mathrm{Ann}(f)$ and $c(f) = (r_1, \ldots, r_m)$. It is easy to see that $Q_i \cap R = \mathrm{Ann}(c(f)) \subseteq \mathrm{Ann}(r_1) \subseteq Z(R)$ and by the Prime Avoidance Theorem, $Q_1 \cap R = \mathrm{Ann}(r_1)$. □

REMARK 2.3.3. Let R be a ring and consider the following three conditions on R:

(1) R is a Noetherian ring.
(2) R has very few zero-divisors.
(3) R has few zero-divisors.

Then, $(1) \rightarrow (2) \rightarrow (3)$ and none of the implications is reversible.

PROOF. For $(1) \rightarrow (2)$ use [**K**, p. 55]. It is obvious that $(2) \rightarrow (3)$.

Suppose k is a field, $A = k[X_1, X_2, X_3, \ldots, X_n, \ldots]$ and $\mathbf{m} = (X_1, X_2, X_3, \ldots, X_n, \ldots)$ and at last $\mathbf{a} = (X_1^2, X_2^2, X_3^2, \ldots, X_n^2, \ldots)$. Since A is a content k-algebra and k has very few zero-divisors, A has very few zero-divisors while it is not a Noetherian ring. Also consider the ring $R = A/\mathbf{a}$. It is easy to check that R is a quasi-local ring with the only prime ideal $\mathbf{m}/\mathbf{a}$ and $Z(R) = \mathbf{m}/\mathbf{a}$ and finally $\mathbf{m}/\mathbf{a} \notin \mathrm{Ass}_R(R)$. Note that $\mathrm{Ass}_R(R) = \emptyset$. □

Now we bring the following definition from [**HK**] and prove some other results for content algebras.

DEFINITION 2.3.4. A ring R has *Property (A)*, if each finitely generated ideal $I \subseteq Z(R)$ has a nonzero annihilator.

Let R be a ring. If R has very few zero-divisors (for example if R is Noetherian), then R has Property (A) [**K**, Theorem 82, p. 56], but there are some non-Noetherian rings which do not have Property (A) [**K**, Exercise 7, p. 63]. The class of non-Noetherian rings having Property (A) is quite large [**Huc**, p. 2].

THEOREM 2.3.5. *Let B be a content R-algebra such that R has Property (A). Then $T(B)$ is a content $T(R)$-algebra, where by $T(R)$, we mean total quotient ring of R.*

PROOF. Let $S' = B - Z(B)$. If $S = S' \cap R$, then $S = R - Z(R)$. We prove that if $c(f) \cap S = \emptyset$, then $f \notin S'$. In fact when $c(f) \cap S = \emptyset$, then $c(f) \subseteq Z(R)$ and since R has Property (A), $c(f)$ has a nonzero annihilator. This means that f is a zero-divisor of B and according to [**OR**, Theorem 6.2, p. 64] the proof is complete. □

THEOREM 2.3.6. *Let B be a content R-algebra such that the content function $c : B \longrightarrow \mathrm{FId}(R)$ is onto, where by $\mathrm{FId}(R)$, we mean the set of finitely generated ideals of R. The following statements are equivalent:*

(1) *R has Property (A).*
(2) *For all $f \in B$, f is a regular element of B iff $c(f)$ is a regular ideal of R.*

PROOF. $(1) \rightarrow (2)$: Let R have Property (A). If $f \in B$ is regular, then for all nonzero $r \in R$, $rf \neq 0$ and so for all nonzero $r \in R$, $rc(f) \neq (0)$, i.e. $\mathrm{Ann}(c(f)) = (0)$ and according to the definition of Property (A), $c(f) \nsubseteq Z(R)$. This means that $c(f)$ is a regular ideal of R. Now let $c(f)$ be a regular ideal of R, so $c(f) \nsubseteq Z(R)$ and therefore $\mathrm{Ann}(c(f)) = (0)$. This means that for all nonzero $r \in R$, $rc(f) \neq (0)$, hence for all nonzero $r \in R$, $rf \neq 0$. Since B is a content R-algebra, f is not a zero-divisor of B.

$(2) \rightarrow (1)$: Let I be a finitely generated ideal of R such that $I \subseteq Z(R)$. Since the content function $c : B \longrightarrow \mathrm{FId}(R)$ is onto, there exists an $f \in B$ such that $c(f) = I$. But $c(f)$ is not a regular ideal of R, therefore according to our assumption, f is not a regular element of B. Since B is a content R-algebra, there exists a nonzero $r \in R$ such that $rf = 0$ and this means that $rI = (0)$, i.e. I has a nonzero annihilator. □

REMARK 2.3.7. In the above theorem the surjectivity condition for the content function c is necessary, because obviously R is a content R-algebra and the condition (2) is satisfied, while one can choose the ring R such that it does not have Property (A) [**K**, Exercise 7, p. 63].

THEOREM 2.3.8. *Let R have property (A) and B be a content R-algebra. Then $Z(B)$ is a finite union of prime ideals in $\mathrm{Min}(B)$ iff $Z(R)$ is a finite union of prime ideals in $\mathrm{Min}(R)$.*

PROOF. The proof is similar to the proof of Theorem 2.3.2 by considering Theorem 2.2.2. □

Please note that if R is a Noetherian reduced ring, then $Z(R)$ is a finite union of prime ideals in $\mathrm{Min}(R)$ (Refer to [**K**, Theorem 88, p. 59] and [**Huc**, Corollary 2.4]).

It is well-known that a ring R has few zero-divisors iff its total quotient ring $T(R)$ is semi-local [**Dav**]. We may suppose that $Z(R) = \bigcup_{i=1}^{n} \mathbf{p}_i$ such that $\mathbf{p}_i \nsubseteq \bigcup_{j=1 \wedge j \neq i}^{n} \mathbf{p}_j$ for all $1 \leq i \leq n$. Then we have $\mathbf{p}_i \nsubseteq \mathbf{p}_j$ for all $i \neq j$ and by the Prime Avoidance Theorem, these prime ideals are uniquely determined. In such a case, it is easy to see that $\mathrm{Max}(T(R)) = \{\mathbf{p}_1 T(R), \ldots, \mathbf{p}_n T(R)\}$, where by $T(R)$ we mean total quotient ring of R. This is the base for the following definition.

DEFINITION 2.3.9. A ring R is said to have *few zero-divisors of degree n*, if R has few zero-divisors and $n = \mathrm{card}\,\mathrm{Max}(T(R))$. In such a case, we write $\mathrm{zd}(R) = n$.

REMARK 2.3.10. If R_i is a ring having few zero-divisors of degree k_i for all $1 \leq i \leq n$, then $\mathrm{zd}(R_1 \times \cdots \times R_n) = \mathrm{zd}(R_1) + \cdots + \mathrm{zd}(R_n)$.

Now we give the following theorem:

THEOREM 2.3.11. *Let B be a content R-algebra. Then the following statements hold for all natural numbers n:*

(1) *If* $\mathrm{zd}(B) = n$*, then* $\mathrm{zd}(R) \leq n$.
(2) *If R has Property (A) and* $\mathrm{zd}(R) = n$*, then* $\mathrm{zd}(B) = n$.
(3) *If the content function* $c : B \longrightarrow \mathrm{FId}(R)$ *is onto, where by* $\mathrm{FId}(R)$*, we mean the set of finitely generated ideals of R, then* $\mathrm{zd}(B) = n$ *iff* $\mathrm{zd}(R) = n$ *and R has Property (A).*

PROOF. (1): Let $Z(B) = \bigcup_{i=1}^{n} Q_i$. We prove that $Z(R) = \bigcup_{i=1}^{n}(Q_i \cap R)$. In order to do that let $r \in Z(R)$. Since $Z(R) \subseteq Z(B)$, there exists an i such that $r \in Q_i$ and therefore $r \in Q_i \cap R$. Now let $r \in Q_i \cap R$ for some i, then $r \in Z(B)$, and this means that there exists a nonzero $g \in B$ such that $rg = 0$ and at last $rc(g) = 0$. Choose a nonzero $d \in c(g)$ and we have $rd = 0$.

(2): Note that with a similar proof given in Theorem 2.3.2, if $Z(R) = \bigcup_{i=1}^{n} \mathbf{p}_i$, then $Z(B) = \bigcup_{i=1}^{n} \mathbf{p}_i B$. Also it is obvious that $\mathbf{p}_i B \subseteq \mathbf{p}_j B$ iff $\mathbf{p}_i \subseteq \mathbf{p}_j$ for all $1 \leq i, j \leq n$. These two imply that $\mathrm{zd}(B) = n$.

3: $(\leftarrow)$ is nothing but (2). For proving $(\rightarrow)$, consider that by (1), we have $\mathrm{zd}(R) \leq n$. Now we prove that ring R has Property (A). Let $I \subseteq Z(R)$ be a finite ideal of R. Choose $f \in B$ such that $I = c(f)$. So $c(f) \subseteq Z(R)$ and by the Prime Avoidance Theorem and (1), there exists an $1 \leq i \leq n$ such that $c(f) \subseteq Q_i \cap R$. Therefore $f \in (Q_i \cap R)B$. But $(Q_i \cap R)B \subseteq Q_i$. So $f \in Z(B)$ and according to McCoy's property for content algebras, there exists a nonzero $r \in R$ such that $f.r = 0$. This means that $I.r = 0$ and I has a nonzero annihilator. Now by (2), we have $\mathrm{zd}(R) = n$. □

COROLLARY 2.3.12. *Let R be a ring and S a commutative, cancellative, torsion-free monoid. Then for all natural numbers n,* $\mathrm{zd}(R[S]) = n$ *iff* $\mathrm{zd}(R) = n$ *and R has Property (A) .*

DEFINITION 2.3.13. An element r of a ring R is said to be *prime to an ideal I* of R if $I : (r) = I$, where by $I : (r)$, we mean the set of all elements c of R such that $cr \in I$ ([**ZS**, p. 223]).

THEOREM 2.3.14. *Let R be a ring, I an ideal of R and B a content R-algebra. Then $f \in B$ is not prime to IB iff $f.r \in IB$ for some $r \in R - I$.*

PROOF. If I is an ideal of R and B a content R-algebra, then B/IB is a content (R/I)-algebra. Assume that $f \in B$ is not prime to IB, so there exists $g \in B$ such that $fg \in IB$, while $g \notin IB$. This means that $f + IB$ is a zero-divisor of B/IB and according to McCoy's property, we have $(f + IB)(r + IB) = IB$ for some $r \in R - I$. □

Let I be an ideal of R. We denote the set of all elements of R that are not prime to I by $S(I)$. It is obvious that $r \in S(I)$ iff $r + I$ is a zero-divisor of the quotient ring R/I. The ideal I is said to be primal if $S(I)$ forms an ideal and in such a case, $S(I)$ is a prime ideal of R. A ring R is said to be *primal*, if the zero ideal of R is primal [**Dau**]. It is obvious that R is primal iff $Z(R)$ is an ideal of R. It is easy to check that if $Z(R)$ is an ideal of R, it is a prime ideal and therefore R is primal iff R has few zero-divisors of degree one, i.e. $\mathrm{zd}(R) = 1$.

THEOREM 2.3.15. *Let B be a content R-algebra. Then the following statements hold:*

(1) *If B is primal, then R is primal and $Z(B) = Z(R)B$.*
(2) *If R is primal and has Property (A), then B is primal, has Property (A) and $Z(B) = Z(R)B$.*
(3) *If the content function $c : B \longrightarrow \mathrm{FId}(R)$ is onto, where by $\mathrm{FId}(R)$, we mean the set of finitely generated ideals of R, then B is primal iff R is primal and has Property (A).*

PROOF. (1): Assume that $Z(B)$ is an ideal of B. We show that $Z(R)$ is an ideal of R. For doing that it is enough to show that if $a, b \in Z(R)$, then $a + b \in Z(R)$. Let $a, b \in Z(R)$. Since $Z(R) \subseteq Z(B)$ and $Z(B)$ is an ideal of B, we have $a + b \in Z(B)$. This means that there exists a nonzero $g \in B$ such that $(a + b)g = 0$. Since $g \neq 0$, we can choose $0 \neq d \in c(g)$ and we have $(a + b)d = 0$. Now it is easy to check that $Z(B) = Z(R)B$.

(2): Let R have Property (A) and $Z(R)$ be an ideal of R. We show that $Z(B) = Z(R)B$. Let $f \in Z(B)$, then there exists a nonzero $r \in R$ such that $f.r = 0$. Therefore we have $c(f) \subseteq Z(R)$ and since $Z(R)$ is an ideal of R, $f \in Z(R)B$. Now let $f \in Z(R)B$, then $c(f) \subseteq Z(R)$. Since R has Property (A), $c(f)$ has a nonzero annihilator and this means that f is a zero-divisor in B. So we have already shown that $Z(B)$ is an ideal of B and therefore B is primal. Finally we prove that B has Property (A). Assume that $J = (f_1, f_2, \ldots, f_n) \subseteq Z(B)$. Therefore $c(f_1), c(f_2), \ldots, c(f_n) \subseteq Z(R)$. But $Z(R)$ is an ideal of R and $c(f_i)$ is a finitely generated ideal of R for any $1 \leq i \leq n$, so $I = c(f_1) + c(f_2) + \cdots + c(f_n) \subseteq Z(R)$ is a finitely generated ideal of R and there exists a nonzero $s \in R$ such that $sI = 0$. This causes $sJ = 0$ and J has a nonzero annihilator in B.

(3): We just need to prove that if B is primal, then R has Property (A). For doing that let $I \subseteq Z(R)$ be a finitely generated ideal of R. Since the content function is onto, there exists an $f \in B$ such that $I = c(f)$. Since $c(f) \subseteq Z(R)$, $f \in Z(B)$. According to McCoy's property for content algebras, we have $f.r = 0$ for some nonzero $r \in R$ and this means $I = c(f)$ has a nonzero annihilator and the proof is complete. □

2.4. Zero-divisor graph of content algebras

Let R be a commutative ring with identity and proper zero-divisors, where by a proper zero-divisor we mean a zero-divisor different from zero. We let $Z(R)^*$ denote the set of proper zero-divisors of R. We consider the graph $\Gamma(R)$, called zero-divisor graph of R, whose vertices are the elements of $Z(R)^*$ and edges are those pairs of distinct proper zero-divisors $\{a, b\}$ such that $ab = 0$.

Recall that a graph is said to be connected if for each pair of distinct vertices v and w, there is a finite sequence of distinct vertices $v = v_1, v_2, \ldots, v_n = w$ such that each pair $\{v_i, v_{i+1}\}$ is an edge. Such a sequence is said to be a path and the distance, $d(v, w)$, between connected vertices v and w is the length of the shortest path connecting them. The diameter of a connected graph G is the supremum of the distances between vertices and is denoted by $\operatorname{diam}(G)$. In [**AL**], zero-divisor graphs were studied and among many things, it was proved that any zero-divisor graph, $\Gamma(R)$, is connected with $0 \leq \operatorname{diam}(\Gamma(R)) \leq 3$ [**AL**, Theorem 2.3]. Note that the diameter is 0 if the graph consists of a single vertex and a connected graph with more than one vertex has diameter 1 if and only if it is complete; i.e., each pair of distinct vertices forms an edge.

In this section, we examine the preservation of diameter of zero-divisor graph under content extensions. What we do is the generalization of what it has been done for polynomial rings in [**ACS**] and [**L**]. The following lemmas are straightforward, but we bring them only for the sake of reference.

LEMMA 2.4.1. *Let R be a ring and B a content R-algebra. Then the following statements hold:*

(1) $\mathrm{Nil}(B) = \mathrm{Nil}(R)B$, *where by* $\mathrm{Nil}(R)$ *we mean all nilpotent elements of R.*
(2) $Z(R) \subseteq Z(B) \subseteq Z(R)B$.
(3) $Z(R)^n = (0)$ *iff* $Z(B)^n = (0)$ *for all* $n \geq 1$.

PROOF. It is well-known that the set of all nilpotent elements of a ring is equal to the intersection of all its minimal primes. For proving (1), use Theorem 2.2.2 and [**OR**, 1.2, p. 51]. The statements (2) is obvious by definition of content modules and [**OR**, 6.1]. For (3), suppose that $Z(R)^n = 0$. Choose $f_1, f_2, \dots, f_n \in Z(B)$. Therefore by McCoy's property for content algebras, $c(f_1), c(f_2), \dots, c(f_n) \subseteq Z(R)$. But by [**Ru**, Proposition 1.1] $c(f_1 f_2 \dots f_n) \subseteq c(f_1)c(f_2)\dots c(f_n) \subseteq Z(R)^n = (0)$. Hence $f_1 f_2 \dots f_n = 0$. □

LEMMA 2.4.2. *Let R be a ring and B a content R-algebra. Then* $\mathrm{diam}(\Gamma(R)) \leq \mathrm{diam}(\Gamma(B))$.

PROOF. Note that the defining homomorphism of R into B is injective and therefore we can suppose R to be a subring of B [**OR**, Remark 6.1(b)] and so $Z(R)^* \subseteq Z(B)^*$. It is obvious that if $\mathrm{diam}(\Gamma(R)) = 0, 1$, or 2, then no path in $\Gamma(R)$ can have a shortcut in $\Gamma(B)$. Now let $\mathrm{diam}(\Gamma(R)) = 3$ and $a - b - c - d$ be the path in $\Gamma(R)$ with $a, b, c, d \in Z(R)^*$ without having any shortcut. Our claim is that neither is there a shortcut for this path in $Z(B)^*$. On the contrary suppose that there is an $h \in Z(B)^*$ such that $a - h - d$ is a path in $Z(B)^*$. Then $a.c(h) = d.c(h) = (0)$. Since $h \neq 0$, there exists a nonzero element $r \in c(h)$ such that $ar = rd = 0$ and this means that $a - r - d$ is a shortcut in $\Gamma(R)$, a contradiction. Therefore $\mathrm{diam}(\Gamma(B)) = 3$. This means that in any case the inequality $\mathrm{diam}(\Gamma(R)) \leq \mathrm{diam}(\Gamma(B))$ holds. □

Recall that $\mathrm{diam}(\Gamma(R)) = 0$ iff R is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[y]/(y^2)$ [**AL**, Example 2.1] and $\mathrm{diam}(\Gamma(R)) = 1$ iff $xy = 0$ for each pair of distinct zero-divisors of R and R has at least two proper zero-divisors [**AL**, Theorem 2.8]. Also from [**L**, Theorem 2.6(3)], we know that $\mathrm{diam}(\Gamma(R)) = 2$ iff either (i) R is reduced with exactly two minimal primes and at least three proper zero-divisors or (ii) $Z(R)$ is an ideal whose square is not (0) and each pair of distinct zero-divisors has a nonzero annihilator. These facts help us to examine the preservation of diameter of zero-divisor graph under content extensions for the cases $\mathrm{diam}(\Gamma(R)) = 0, 1$.

THEOREM 2.4.3. *Let R be a ring and B be a content R-algebra and $B \ncong R$. Then the following statements hold:*

(1) $\mathrm{diam}(\Gamma(R)) = 0$ *and* $\mathrm{diam}(\Gamma(B)) = 1$ *iff either* $R \cong \mathbb{Z}_4$ *or* $R \cong \mathbb{Z}_2[y]/(y^2)$.

(2) $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(B)) = 1$ *iff R is a nonreduced ring with more than one proper zero-divisor and $Z(R)^2 = 0$.*

(3) $\operatorname{diam}(\Gamma(R)) = 1$ *and* $\operatorname{diam}(\Gamma(B)) = 2$ *iff* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

PROOF. Let B be a content R-algebra such that $B \ncong R$. For (1), we just need to prove that if $\operatorname{diam}(\Gamma(R)) = 0$, then $\operatorname{diam}(\Gamma(B)) = 1$. It is obvious that R is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[y]/(y^2)$ [**AL**, Example 2.1]. But $Z(\mathbb{Z}_4) = (2)$ and $Z(\mathbb{Z}_2[y]/(y^2)) = \{by+(y^2) : b \in \mathbb{Z}_2\} = (y)$, so in any case $Z(B) = Z(R)B$ by Theorem 2.3.15. It is easy to check that $fg = 0$ for any distinct pair of zero-divisors f, g in B and B has at least two proper zero-divisors. So according to [**L**, Theorem 2.6(2)], $\operatorname{diam}(\Gamma(B)) = 1$.

(2) and (3): If R is a nonreduced ring with more than one proper zero-divisor and $Z(R)^2 = (0)$ then $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $ab = 0$ for all $a, b \in Z(R)$ [**AL**, Theorem 2.8]. This means that $\operatorname{diam}(\Gamma(R)) = 1$. Let $f, g \in Z(B)^*$. According to McCoy's property for content algebras, there exist nonzero $r, s \in R$ such that $c(f).r = c(g).s = 0$. This implies that $c(f) \subseteq Z(R)$ and $c(g) \subseteq Z(R)$ and therefore $c(fg) \subseteq c(f)c(g) = (0)$. But B has at least two proper zero-divisors, since $Z(R)^* \subseteq Z(B)^*$. Hence $\operatorname{diam}(\Gamma(B)) = 1$.

Now let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then R is a reduced ring with exactly two minimal prime ideals, $\mathbf{p}$ and $\mathbf{q}$, where $\mathbf{p} = ((1,0))$ and $\mathbf{q} = ((0,1))$ and according to Lemma 2.4.1 and Corollary 2.2.2, B is a reduced ring with exactly two minimal prime ideals, $\mathbf{p}B$ and $\mathbf{q}B$. It is obvious that B has at least two proper zero-divisors. If B has exactly two proper zero-divisors, then $Z(B)^* = \{(1,0),(0,1)\}$ and therefore according to [**AL**, Theorem 2.8], $B \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore B has at least three proper zero-divisors and according to [**L**, Theorem 2.6(3)], we have $\operatorname{diam}(\Gamma(B)) = 2$. By this discussion, it is, then, obvious that $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(B)) = 1$ implies R is a nonreduced ring with more than one proper zero-divisor and $Z(R)^2 = 0$.

Now let $\operatorname{diam}(\Gamma(R)) = 1$ and $\operatorname{diam}(\Gamma(B)) = 2$. If $Z(R)^2 = (0)$, then $Z(B)^2 = 0$ and $\operatorname{diam}(\Gamma(B)) = 1$, therefore $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ by [**AL**, Theorem 2.8]. □

Now we examine the preservation of diameter of zero-divisor graph under content extensions with $\operatorname{diam}(\Gamma(R)) = 2$.

THEOREM 2.4.4. *Let B be a content R-algebra such that the content function $c : B \longrightarrow$* $\operatorname{FId}(R)$ *is onto, where by* $\operatorname{FId}(R)$*, we mean the set of finitely generated ideals of R. Let R has at least three proper zero-divisors and $B \ncong R$. Then the following statements hold:*

(1) $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(B)) = 2$ *iff either (i) R is a reduced ring with exactly two minimal prime ideals and R has more than two proper zero-divisors, or (ii) R is a primal ring with $Z(R)^2 \neq (0)$ and R has Property (A).*

(2) $\operatorname{diam}(\Gamma(R)) = 2$ *and* $\operatorname{diam}(\Gamma(B)) = 3$ *iff $Z(R)$ is an ideal of R and R does not have Property (A) but each pair of proper zero-divisors of R has a nonzero annihilator.*

PROOF. (1): If R is a reduced ring with exactly two minimal prime ideals and R has more than two proper zero-divisors, then according to Lemma 2.3.1 and Corollary 2.1.3, B is a reduced ring with exactly two minimal prime ideals and obviously B has more than two proper zero-divisors and therefore according to [**L**, Theorem 2.6(3)], $\mathrm{diam}(\Gamma(R)) = \mathrm{diam}(\Gamma(B)) = 2$. If R is a primal ring with $Z(R)^2 \neq (0)$ and R has Property (A), then by Theorem 2.3.15, B is primal and has Property (A). Also obviously $Z(B)^2 \neq 0$. So according to [**L**, Theorem 2.6(3)], $\mathrm{diam}(\Gamma(R)) = \mathrm{diam}(\Gamma(B)) = 2$. Now let $\mathrm{diam}(\Gamma(R)) = \mathrm{diam}(\Gamma(B)) = 2$. If R is a reduced ring with exactly two minimal prime ideals and R has more than two proper zero-divisors, then we are done, otherwise, $Z(B)$ is an ideal whose square is not (0) and each pair of distinct zero-divisors has a nonzero annihilator. Since $Z(B)$ is primal, then $Z(R)$ is an ideal of R and R has Property (A) by Theorem 2.3.15. But $Z(B)^2 \neq 0$ implies that $Z(R)^2 \neq 0$ and the proof is complete.

(2) Assume that $Z(R)$ is an ideal of R and R does not have Property (A) but each pair of proper zero-divisors of R have a nonzero annihilator. It is obvious that $\mathrm{diam}(\Gamma(R)) = 2$. Our claim is that $\mathrm{diam}(\Gamma(B)) = 3$. On the contrary, let $\mathrm{diam}(\Gamma(B)) = 2$. According to [**L**, Theorem 2.6(3)] and [**Huc**, Corollary 2.4], either $\mathrm{zd}(R) = 2$ or $\mathrm{zd}(R) = 1$. But the content function $c : B \longrightarrow \mathrm{FId}(R)$ is onto and in both cases, by Theorem 2.2.11, R has Property (A), a contradiction. Therefore $\mathrm{diam}(\Gamma(B)) = 3$. Now let $\mathrm{diam}(\Gamma(R)) = 2$ and $\mathrm{diam}(\Gamma(B)) = 3$, then according to [**L**, Theorem 2.6(3)] and Theorem 2.1.15, $Z(R)$ is an ideal of R and each pair of proper zero-divisors of R has a nonzero annihilator. Obviously R does not have Property (A), otherwise $\mathrm{diam}(\Gamma(B)) = 2$ and the proof is complete. □

Note that the two recent theorems are the generalization of [**L**, Theorem 3.6]. Consider that in the last theorem, we assume the content function $c : B \longrightarrow \mathrm{FId}(R)$ to be onto. In the following we state a theorem similar to [**ACS**, Proposition 5], without assuming the content function $c : B \longrightarrow \mathrm{FId}(R)$ to be onto.

THEOREM 2.4.5. *Let B be a content R-algebra and $Z(R)^n = (0)$, while $Z(R)^{n-1} \neq (0)$ for some $n \geqq 2$. Then the following statements hold:*

(1) *If $n = 2$, then* $\mathrm{diam}(\Gamma(R)) = \mathrm{diam}(\Gamma(B)) = 1$.
(2) *If $n > 2$, then* $\mathrm{diam}(\Gamma(R)) = \mathrm{diam}(\Gamma(B)) = 2$.

PROOF. (1) holds by [**AL**, Theorem 2.8] and Theorem 2.2.3(3).

(2): By assumption $Z(R)^n = (0)$ and $Z(R)^2 \neq (0)$. Therefore $\Gamma(R)$ is not a complete graph and so there exist distinct $a, b \in Z(R)^*$ such that $ab \neq 0$. Since $Z(R)^{n-1} \neq (0)$, there exist $c_1, c_2, \ldots, c_{n-1} \in Z(R)$ such that $c = c_1c_2 \ldots c_{n-1} \neq 0$. So $c \neq a, b$ and $ca = cb = 0$. Hence $\mathrm{diam}(\Gamma(R)) = 2$. On the other hand $Z(R)^{n-1} \neq (0)$ causes $Z(B)^{n-1} \neq (0)$. By Lemma 2.3.1, $Z(B)^n = (0)$. This means that $\mathrm{diam}(\Gamma(B)) = 2$ and the proof is complete. □

2.5. Gaussian and Armendariz algebras

Let B be an R-algebra that is a content R-module. Suppose $f \in B$. Therefore $f \in c(f)B$, since B is a content R-module. This means that $f = \sum a_i f_i$, where $a_i \in R$ and $f_i \in B$ and $c(f) = (a_1, a_2, \ldots, a_n)$. Similarly if $g \in B$, then $g = \sum b_j g_j$, where $b_j \in R$ and $g_j \in B$ and $c(g) = (b_1, b_2, \ldots, b_m)$. Then $fg = \sum a_i b_j f_i g_j \in c(f)c(g)B$, and hence $c(fg) \subseteq c(f)c(g)$ [**Ru**, Proposition 1.1, p. 330]. The question that when the equality holds, is the base for the following definition:

DEFINITION 2.5.1. Let B be an R-algebra that is a content R-module. B is said to be a *Gaussian R-algebra* if $c(fg) = c(f)c(g)$, for all $f, g \in B$.

For example if B is a content R-algebra such that every nonzero finitely generated ideal of R is a cancellation ideal of R, then B is a Gaussian R-algebra. Another example is given in the following remark:

REMARK 2.5.2. Let $(R, \mathbf{m})$ be a quasi-local ring with $\mathbf{m}^2 = (0)$. If B is a content R-algebra, then B is a Gaussian R-algebra.

PROOF. Let $f, g \in B$ such that $c(f) \subseteq \mathbf{m}$ and $c(g) \subseteq \mathbf{m}$, then $c(fg) = c(f)c(g) = (0)$, otherwise one of them, say $c(f)$, is R and according to Dedekind-Mertens content formula, we have $c(fg) = c(g) = c(f)c(g)$. □

THEOREM 2.5.3. *Let M be an R-module such that each finitely generated submodule of M is cyclic and S be a commutative, cancellative, torsion-free monoid. Then for all $f \in R[S]$ and $g \in M[S]$, we have: $c(fg) = c(f)c(g)$.*

PROOF. Let $g = m_1 X^{g_1} + m_2 X^{g_2} + \cdots + m_n X^{g_n}$, where $m_i \in M$ and $g_i \in S$ for all $0 \leq i \leq n$. Then there exists an $m \in M$, such that $c(g) = (m_1, m_2, \ldots, m_n) = (m)$. From this we have $m_i = r_i m$ and $m = \sum s_i m_i$, where $r_i, s_i \in R$. Put $d = \sum s_i r_i$, then $m = dm$. Since S is an infinite set, it is possible to choose $g_{n+1} \in S - \{g_1, g_2, \ldots, g_n\}$.

Now put $g' = r_1 X^{g_1} + r_2 X^{g_2} + \cdots + r_n X^{g_n} + (1-d) X^{g_{n+1}}$. One can easily check that $g = g'm$ and $c(g') = R$ and $c(fg) = c(fg'm) = c(fg')m = c(f)m = c(f)c(g)$. □

COROLLARY 2.5.4. *Let R be a ring such that every finitely generated ideal of R is principal and S be a commutative, cancellative, torsion-free monoid. Then $R[S]$ is a Gaussian R-algebra.*

For more about content formulas for polynomial modules, one can refer to [**NY**] and [**AK**]. In the next step, we define Armendariz algebras and show their relationships with Gaussian algebras. Armendariz rings were introduced in [**RC**]. A ring R is said to be an Armendariz ring if for all $f, g \in R[X]$ with $f = a_0 + a_1 X + \cdots + a_n X^n$ and $g = b_0 + b_1 X +$

$\cdots + b_m X^m$, $fg = 0$ implies $a_i b_j = 0$, for all $0 \le i \le n$ and $0 \le j \le m$. This is equivalent to say that if $fg = 0$, then $c(f)c(g) = 0$ and our inspiration to define Armendariz algebras.

DEFINITION 2.5.5. Let B be an R-algebra such that it is a content R-module. We say B is an *Armendariz R-algebra* if for all $f, g \in B$, if $fg = 0$, then $c(f)c(g) = (0)$.

For example if B is a weak content R-algebra and R is a reduced ring, then B is an Armendariz R-algebra.

THEOREM 2.5.6. *Let R be a ring and (0) a $\mathbf{p}$-primary ideal of R such that $\mathbf{p}^2 = (0)$ and B a content R-algebra. Then B is an Armendariz R-algebra.*

PROOF. Let $f, g \in B$, where $fg = 0$. If $f = 0$ or $g = 0$, then definitely $c(f)c(g) = 0$, otherwise suppose that $f \neq 0$ and $g \neq 0$, therefore f and g are both zero-divisors of B. Since (0) is a $\mathbf{p}$-primary ideal of R, so (0) is a $\mathbf{p}B$-primary ideal of B [**Ru**, p. 331] and therefore $\mathbf{p}B$ is the set of zero-divisors of B. So $f, g \in \mathbf{p}B$ and this means that $c(f) \subseteq \mathbf{p}$ and $c(g) \subseteq \mathbf{p}$. Finally $c(f)c(g) \subseteq \mathbf{p}^2 = (0)$. □

In order to characterize Gaussian algebras in terms of Armendariz algebras, we should mention the following useful remark.

REMARK 2.5.7. Let R be a ring and I an ideal of R. If B is a Gaussian R-algebra then B/IB is a Gaussian (R/I)-algebra as well.

THEOREM 2.5.8. *Let B be a content R-algebra. Then B is a Gaussian R-algebra iff for any ideal I of R, B/IB is an Armendariz (R/I)-algebra.*

PROOF. $(\rightarrow)$: According to the above remark, since B is a Gaussian R-algebra, B/IB is a Gaussian (R/I)-algebra and obviously any Gaussian algebra is an Armendariz algebra and this completes the proof.

$(\leftarrow)$: One can easily check that if B is an algebra such that it is a content R-module, then for all $f, g \in B$, $c(fg) \subseteq c(f)c(g)$ [**Ru**, Proposition 1.1, p. 330]. Therefore we need to prove that $c(f)c(g) \subseteq c(fg)$. Put $I = c(fg)$, since B/IB is an Armendariz (R/I)-algebra and $c(fg + IB) = I$ so $c(f + IB)c(g + IB) = I$ and this means that $c(f)c(g) \subseteq c(fg)$. □

For more about Armendariz and Gaussian rings, one can refer to [**AC**].

2.6. The content algebra $l(R, B)$

Let B be a content R-algebra and $S' = \{f \in B \colon c(f) = R\}$. It is easy to check that $S' = B - \bigcup_{\mathbf{m} \in \mathrm{Max}(R)} \mathbf{m}B$ and $S = S' \cap R = U(R)$, where by $U(R)$, we mean the units of R. According to [**OR**, Theorem 6.2, p. 64], it is clear that $l(R, B) = B_{S'}$ is also a content R-algebra and B is a subring of $l(R, B)$. The content algebra $l(R, B)$ is, in fact, the generalization of the concept of the ring $R(X)$.

DEFINITION 2.6.1. A ring R is called *presimplifiable* if any zero-divisor of R is a element of the Jacobson radical of R, i.e. $Z(R) \subseteq \mathrm{Jac}(R)$[**AFS**].

THEOREM 2.6.2. *Let B be a content R-algebra such that $S' = \{f \in B \colon c(f) = R\}$ and put $R' = l(R,B) = B_{S'}$, then the following statements hold:*

(1) *The map $\varphi : \mathrm{Max}(R) \longrightarrow \mathrm{Max}(R')$, defined by $I \longrightarrow IR'$ is a bijection.*
(2) $\mathrm{Jac}(R') = \mathrm{Jac}(R)R'$.
(3) $U(R') = \{f/g \colon c(f) = c(g) = R\}$
(4) *The ring R' is presimplifiable iff R is presimplifiable.*
(5) *The ring R' is 0-dimensional iff R is 0-dimensional.*

PROOF. The first proposition is actually a special case of [**G1**, 4.8]. For the proof of the second proposition notice that the Jacobson radical of a ring is the intersection of all maximal ideals. Now use [**OR**, 1.2, p. 51].

It is obvious that if $c(f) = c(g) = R$, then f/g is a unit of R'. Now let f/g be a unit of R', where $c(g) = R$ and assume that there exists a element of R', say f'/g' with $c(g') = R$, such that $(f/g).(f'/g') = 1/1$. According to McCoy's property for content algebras, $S' \subseteq B - Z_B(B)$. So $ff' = gg'$ and $ff' \in S'$. This means that $f \in S'$ and the proof of the third proposition is complete.

For the proof of the forth proposition, suppose R is presimplifiable and let $f \in Z(R')$. Therefore there exists a nonzero $r \in R$ such that $rf = 0$ and so $rc(f) = (0)$. This means that $c(f) \subseteq Z(R)$. Since R is presimplifiable, $c(f) \subseteq \mathrm{Jac}(R)$ and at last $f \in \mathrm{Jac}(R)R'$ and according to (2) $f \in \mathrm{Jac}(R')$. It is easy to check that if R' is presimplifiable then R is presimplifiable as well. For the proof of the fifth proposition note that a ring, say T, is 0-dimensional iff $\mathrm{Min}(T) = \mathrm{Max}(T)$. □

THEOREM 2.6.3. *Let B be a content R-algebra with the property that if $f \in B$ with $c(f) = (a)$ where $a \in R$, then there exists an $f_1 \in B$ such that $f = af_1$ and $c(f_1) = R$ and put $S' = \{f \in B \colon c(f) = R\}$ and $R' = l(R,B) = B_{S'}$. Then the idempotent elements of R and R' coincide.*

PROOF. Let f/g be an idempotent element of R', where $f, g \in B$ and $c(g) = R$. Therefore $fg^2 = gf^2$ and since g is a regular element of B, we have $fg = f^2$. So $c(f^2) = c(fg) = c(f)$, but $c(f^2) \subseteq c(f)^2$, therefore $c(f)^2 = c(f)$. We know that every finitely generated idempotent ideal of a ring is generated by an idempotent element of the ring [**G1**, p. 63]. Therefore we can suppose that $c(f) = (e)$ such that $e^2 = e$. On the other side we can find an $f_1 \in B$ such that $f = ef_1$ and $c(f_1) = R$. Consider $ef_1/g = f/g = f^2/g^2 = e^2 f_1^2/g^2$. Since f_1 and g are both regular, and e is idempotent, we have $e = ef_1/g = f/g \in R$. □

COROLLARY 2.6.4. *Let R be a ring and M a commutative, cancellative and torsion-free monoid and put $S' = \{f \in R[M] : c(f) = R\}$ and $R' = B_{S'}$. Then the idempotent elements of R and R' coincide.*

DEFINITION 2.6.5. A commutative ring R is said to be a *valuation ring* if for any a and b in R either a divides b or b divides a [**K**, p. 35].

THEOREM 2.6.6. *Let B be a content R-algebra with the property that if $f \in B$ with $c(f) = (a)$ where $a \in R$, then there exists an $f_1 \in B$ such that $f = af_1$ and $c(f_1) = R$ and put $S' = \{f \in B\colon c(f) = R\}$ and $R' = l(R,B) = B_{S'}$. If R is a valuation ring, then so is $R' = B_{S'}$.*

PROOF. Let f/g be a element of R', where $f, g \in B$ and $c(g) = R$. Since $c(f)$ is a finitely generated ideal of R and R is a valuation ring, there exists an $r \in R$ such that $c(f) = (r)$ and therefore there exists an $f_1 \in B$ such that $f = rf_1$ and $c(f_1) = R$. By considering this fact that f_1/g is a unit in R', it is obvious that R' is also a valuation ring and the proof is complete. □

2.7. Zero-divisors of semigroup modules

Let us recall that if R is a ring and $f = a_0 + a_1X + \cdots + a_nX^n$ is a polynomial on the ring R, then content of f is defined as the R-ideal, generated by the coefficients of f, i.e. $c(f) = (a_0, a_1, \ldots, a_n)$. The content of an element of a semigroup module is a natural generalization of the content of a polynomial as follows:

DEFINITION 2.7.1. Let M be an R-module and S be a commutative semigroup. Let $g \in M[S]$ and put $g = m_1X^{s_1} + m_2X^{s_2} + \cdots + m_nX^{s_n}$, where $m_1, \ldots, m_n \in M$ and $s_1, \ldots, s_n \in S$. We define the content of g to be the R-submodule of M generated by the coefficients of g, i.e. $c(g) = (m_1, \ldots, m_n)$.

THEOREM 2.7.2. *Let S be a commutative monoid and M be a nonzero R-module. Then the following statements are equivalent:*

(1) *S is a cancellative and torsion-free monoid.*
(2) *For all $f \in R[S]$ and $g \in M[S]$, there is a natural number k such that $c(f)^k c(g) = c(f)^{k-1} c(fg)$.*
(3) *(McCoy's Property) For all $f \in R[S]$ and $g \in M[S] - \{0\}$, if $fg = 0$, then there exists an $m \in M - \{0\}$ such that $f.m = 0$.*
(4) *For all $f \in R[S]$, $\mathrm{Ann}_M(c(f)) = 0$ if and only if $f \notin Z_{R[S]}(M[S])$.*

PROOF. $(1) \to (2)$ has been proved in [**No**].

For $(2) \to (3)$, assume that $f \in R[S]$ and $g \in M[S] - \{0\}$, such that $fg = 0$. So there exists a natural number k such that $c(f)^k c(g) = c(f)^{k-1} c(fg) = (0)$. Take t the smallest natural number such that $c(f)^t c(g) = (0)$ and choose m a nonzero element of $c(f)^{t-1}c(g)$. It is easy to check that $f.m = 0$.

For $(3) \to (1)$, we prove that if S is not cancellative or not torsion-free then (1) cannot hold. For the moment, suppose that S is not cancellative, so there exist $s,t,u \in S$ such that $s+t = s+u$ while $t \neq u$. Put $f = X^s$ and $g = (qX^t - qX^u)$, where q is a nonzero element of M. Then obviously $fg = 0$, while $f.m \neq 0$ for all $m \in M - \{0\}$. Finally suppose that S is cancellative but not torsion-free. Let $s,t \in S$ be such that $s \neq t$, while $ns = nt$ for some natural n. Choose the natural number k minimal so that $ks = kt$. Then we have $0 = qX^{ks} - qX^{kt} = (\sum_{i=0}^{k-1} X^{(k-i-1)s+it})(qX^s - qX^t)$, where q is a nonzero element of M.

Since S is cancellative, the choice of k implies that $(k-i_1-1)s + i_1 t \neq (k-i_2-1)s + i_2 t$ for $0 \leq i_1 < i_2 \leq k-1$. Therefore $\sum_{i=0}^{k-1} X^{(k-i-1)s+it} \neq 0$, and this completes the proof. $(3) \leftrightarrow (4)$ is obvious. □

COROLLARY 2.7.3. *Let M be an R-module and S be a commutative, cancellative and torsion-free monoid. Then the following statements hold:*

(1) *R is a domain if and only if $R[S]$ is a domain.*
(2) *If $\boldsymbol{p}$ is a prime ideal of R, then $\boldsymbol{p}[S]$ is a prime ideal of $R[S]$.*
(3) *If $\boldsymbol{p}$ is in $\mathrm{Ass}_R(M)$, then $\boldsymbol{p}[S]$ is in $\mathrm{Ass}_{R[S]}(M[S])$.*

DEFINITION 2.7.4. Let M be an R-module and P be a proper R-submodule of M. P is said to be a *prime submodule* (*primary submodule*) of M, if $rx \in P$ implies $x \in P$ or $rM \subseteq P$ (there exists a natural number n such that $r^n M \subseteq P$), for each $r \in R$ and $x \in M$.

COROLLARY 2.7.5. *Let M be an R-module and S be a commutative, cancellative and torsion-free monoid. Then the following statements hold:*

(1) (0) *is a prime (primary) submodule of M if and only if* (0) *is a prime (primary) submodule of $M[S]$.*
(2) *If P is a prime (primary) submodule of M, then $P[S]$ is a prime (primary) submodule of $M[S]$.*

In [**Dav**], it has been defined that a ring R has *few zero-divisors*, if $Z(R)$ is a finite union of prime ideals. We give the following definition and prove some interesting results about zero-divisors of semigroup modules. Modules having (very) few zero-divisors, introduced in [**Na**], have also some interesting homological properties [**NP**]. In chapter 3, we discuss homological properties of modules having very few zero-divisors.

DEFINITION 2.7.6. An R-module M has *very few zero-divisors*, if $Z_R(M)$ is a finite union of prime ideals in $\mathrm{Ass}_R(M)$.

REMARK 2.7.7. *Examples of modules having very few zero-divisors.* If R is a Noetherian ring and M is an R-module such that $\mathrm{Ass}_R(M)$ is finite, then obviously M has very few zero-divisors. For example $\mathrm{Ass}_R(M)$ is finite if M is a finitely generated R-module [**K**, p. 55]. Also if R is a Noetherian quasi-local ring and M is a balanced big Cohen-Macaulay R-module, then $\mathrm{Ass}_R(M)$ is finite [**BH**, Proposition 8.5.5, p. 344].

Note that Remark 2.2.3 shows that there are rings having few zero-divisors, while not having very few zero-divisors.

THEOREM 2.7.8. *Let M be an R-module and S a commutative, cancellative and torsion-free monoid. Then the $R[S]$-module $M[S]$ has very few zero-divisors, if and only if the R-module M has very few zero-divisors.*

PROOF. ($\leftarrow$): Let $Z_R(M) = \mathbf{p}_1 \cup \mathbf{p}_2 \cup \cdots \cup \mathbf{p}_n$, where $\mathbf{p}_i \in \mathrm{Ass}_R(M)$ for all $1 \leq i \leq n$. We will show that $Z_{R[S]}(M[S]) = \mathbf{p}_1[S] \cup \mathbf{p}_2[S] \cup \cdots \cup \mathbf{p}_n[S]$. Let $f \in Z_{R[S]}(M[S])$, so there exists an $m \in M - \{0\}$ such that $f.m = 0$ and so $c(f).m = (0)$. Therefore $c(f) \subseteq Z_R(M)$ and this means that $c(f) \subseteq \mathbf{p}_1 \cup \mathbf{p}_2 \cup \cdots \cup \mathbf{p}_n$ and according to the Prime Avoidance Theorem, we have $c(f) \subseteq \mathbf{p}_i$, for some $1 \leq i \leq n$ and therefore $f \in \mathbf{p}_i[S]$. Now let $f \in \mathbf{p}_1[S] \cup \mathbf{p}_2[S] \cup \cdots \cup \mathbf{p}_n[S]$, so there exists an i such that $f \in \mathbf{p}_i[S]$, so $c(f) \subseteq \mathbf{p}_i$ and $c(f)$ has a nonzero annihilator in M and this means that f is a zero-divisor of $M[S]$. Note that by Corollary 3, $\mathbf{p}_i[S] \in \mathrm{Ass}_{R[S]}(M[S])$ for all $1 \leq i \leq n$.

($\rightarrow$): Let $Z_{R[S]}(M[S]) = \cup_{i=1}^n Q_i$, where $Q_i \in \mathrm{Ass}_{R[S]}(M[S])$ for all $1 \leq i \leq n$. Therefore $Z_R(M) = \cup_{i=1}^n (Q_i \cap R)$. Without loss of generality, we can assume that $Q_i \cap R \nsubseteq Q_j \cap R$ for all $i \neq j$. Now we prove that $Q_i \cap R \in \mathrm{Ass}_R(M)$ for all $1 \leq i \leq n$. Consider $g \in M[S]$ such that $Q_i = \mathrm{Ann}(g)$ and $g = m_1 X^{s_1} + m_2 X^{s_2} + \cdots + m_n X^{s_n}$, where $m_1, \ldots, m_n \in M$ and $s_1, \ldots, s_n \in S$. It is easy to see that $Q_i \cap R = \mathrm{Ann}(c(g)) \subseteq \mathrm{Ann}(m_1) \subseteq Z_R(M)$ and by the Prime Avoidance Theorem, $Q_1 \cap R = \mathrm{Ann}(m_1)$. □

In [**HK**], it has been defined that a ring R has *Property (A)*, if each finitely generated ideal $I \subseteq Z(R)$ has a nonzero annihilator. We give the following definition:

DEFINITION 2.7.9. An R-module M has *Property (A)*, if each finitely generated ideal $I \subseteq Z_R(M)$ has a nonzero annihilator in M.

REMARK 2.7.10. If the R-module M has very few zero-divisors, then M has Property (A).

THEOREM 2.7.11. *Let S be a commutative, cancellative and torsion-free monoid and M be an R-module. The following statements are equivalent:*

(1) *The R-module M has Property (A).*
(2) *For all $f \in R[S]$, f is $M[S]$-regular if and only if $c(f)$ is M-regular.*

PROOF. (1) $\rightarrow$ (2): Let the R-module M have Property (A). If $f \in R[S]$ is $M[S]$-regular, then $f.m \neq 0$ for all nonzero $m \in M$ and so $c(f).m \neq (0)$ for all nonzero $m \in M$

and according to the definition of Property (A), $c(f) \nsubseteq Z_R(M)$. This means that $c(f)$ is M-regular. Now let $c(f)$ be M-regular, so $c(f) \nsubseteq Z_R(M)$ and this means that $c(f).m \neq (0)$ for all nonzero $m \in M$ and hence $f.m \neq 0$ for all nonzero $m \in M$. Since S is a commutative, cancellative and torsion-free monoid, f is not a zero-divisor of $M[S]$, i.e. f is $M[S]$-regular.

$(2) \rightarrow (1)$: Let I be a finitely generated ideal of R such that $I \subseteq Z_R(M)$. Then there exists an $f \in R[S]$ such that $c(f) = I$. But $c(f)$ is not M-regular, therefore according to our assumption, f is not $M[S]$-regular. Therefore there exists a nonzero $m \in M$ such that $f.m = 0$ and this means that $I.m = (0)$, i.e. I has a nonzero annihilator in M. □

Let, for the moment, M be an R-module such that the set $Z_R(M)$ of zero-divisors of M is a finite union of prime ideals. One can consider $Z_R(M) = \cup_{i=1}^n \mathbf{p}_i$ such that $\mathbf{p}_i \nsubseteq \cup_{j=1 \wedge j \neq i}^n \mathbf{p}_j$ for all $1 \leq i \leq n$. Obviously we have $\mathbf{p}_i \nsubseteq \mathbf{p}_j$ for all $i \neq j$. Also, it is easy to check that, if $Z_R(M) = \cup_{i=1}^n \mathbf{p}_i$ and $Z_R(M) = \cup_{k=1}^m \mathbf{q}_k$ such that $\mathbf{p}_i \nsubseteq \mathbf{p}_j$ for all $i \neq j$ and $\mathbf{q}_k \nsubseteq \mathbf{q}_l$ for all $k \neq l$, then $m = n$ and $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\} = \{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$, i.e. these prime ideals are uniquely determined (Use the Prime Avoidance Theorem). This is the base for the following definition:

DEFINITION 2.7.12. An R-module M is said to have *few zero-divisors of degree n*, if $Z_R(M)$ is a finite union of n prime ideals $\mathbf{p}_1, \ldots, \mathbf{p}_n$ of R such that $\mathbf{p}_i \nsubseteq \mathbf{p}_j$ for all $i \neq j$.

THEOREM 2.7.13. *Let M be an R-module and S a commutative, cancellative and torsion-free monoid. Then the $R[S]$-module $M[S]$ has few zero-divisors of degree n, if and only if the R-module M has few zero-divisors of degree n and Property (A).*

PROOF. $(\leftarrow)$: By considering the R-module M having Property (A), similar to the proof of Theorem 9, we have if $Z_R(M) = \cup_{i=1}^n \mathbf{p}_i$, then $Z_{R[S]}(M[S]) = \cup_{i=1}^n \mathbf{p}_i[S]$. Also it is obvious that $\mathbf{p}_i[S] \subseteq \mathbf{p}_j[S]$ if and only if $\mathbf{p}_i \subseteq \mathbf{p}_j$ for all $1 \leq i, j \leq n$. These two imply that the $R[S]$-module $M[S]$ has few zero-divisors of degree n.

$(\rightarrow)$: Note that $Z_R(M) \subseteq Z_{R[S]}(M[S])$. It is easy to check that if $Z_{R[S]}(M[S]) = \cup_{i=1}^n Q_i$, where Q_i are prime ideals of $R[S]$ for all $1 \leq i \leq n$, then $Z_R(M) = \cup_{i=1}^n (Q_i \cap R)$. Now we prove that the R-module M has Property (A). Let $I \subseteq Z_R(M)$ be a finite ideal of R. Choose $f \in R[S]$ such that $I = c(f)$. So $c(f) \subseteq Z_R(M)$ and obviously $f \in Z_{R[S]}(M[S])$ and according to McCoy's property, there exists a nonzero $m \in M$ such that $f.m = 0$. This means that $I.m = 0$ and I has a nonzero annihilator in M. Consider that by a similar discussion in $(\leftarrow)$, the R-module M has few zero-divisors obviously not less than degree n and this completes the proof. □

An R-module M is said to be *primal*, if $Z_R(M)$ is an ideal of R [**Dau**]. It is easy to check that if $Z_R(M)$ is an ideal of R, then it is a prime ideal and therefore the R-module M is primal if and only if M has few zero-divisors of degree one.

COROLLARY 2.7.14. *Let M be an R-module and S a commutative, cancellative and torsion-free monoid. Then the $R[S]$-module $M[S]$ is primal, if and only if the R-module M is primal and has Property (A).*

CHAPTER 3

Grade of zero-divisor modules

3.1. Modules having very few zero-divisors

Let us, first, recall the concept of modules having very few zero-divisors. In [**Dav**], it has been defined that a ring R has *few zero-divisors* if $Z_R(R)$ is a finite union of prime ideals. Similarly it has been defined in [**Na**] that an R-module M has *very few zero-divisors*, if $Z_R(M)$ is a finite union of prime ideals in $\mathrm{Ass}_R(M)$. In the following we give some properties of modules having very few zero-divisors.

LEMMA 3.1.1. *The following statements hold:*

(1) *Let S be a commutative ring with identity, $f : R \longrightarrow S$ a ring homomorphism, and M an S-module. If M as an S-module has very few zero-divisors, then M as an R-module has very few zero-divisors.*
(2) *Let I be an ideal of R and M an R-module. Then M/IM as an R/I-module has very few zero-divisors if and only if as an R-module has very few zero-divisors.*

PROOF. The proof is easy and left to the reader. □

THEOREM 3.1.2. *Let M be an R-module. Consider the following three conditions:*

(1) *M is Noetherian.*
(2) *M has very few zero-divisors.*
(3) *$Z_R(M)$ is finite union of prime ideals of R.*

Then, $(1) \to (2) \to (3)$ *and none of the implications are reversible.*

PROOF. $(1) \to (2)$ Note that $R/\mathrm{Ann}(M)$ is a Noetherian ring and M is a finitely generated $R/\mathrm{Ann}(M)$-module, so M as an $R/\mathrm{Ann}(M)$-module has very few zero-divisors by Theorems 6.1(ii) and 6.5(i) in [**Ma**]. Hence Lemma 3.1.1(1) shows that M as an R-module has very few zero-divisors.

$(2) \to (3)$ Obvious.

The following examples show that none of the above implications are reversible.

Suppose k is a field, $A = k[X_1, X_2, X_3, \ldots, X_n, \ldots]$, $\mathbf{m} = (X_1, X_2, X_3, \ldots, X_n, \ldots)$ and $I = (X_1^2, X_2^2, X_3^2, \ldots, X_n^2, \ldots)$. Since A is an integral domain, $Z_A(A) = \{0\}$ and $(0) \in \mathrm{Ass}_A(A)$. So A has very few zero-divisors while it is not a Noetherian ring. Let $R = A/I$. It is easy to check that the only prime ideal of R is $\mathbf{m}/I$ and $Z_R(R) = \mathbf{m}/I$. Also $\mathbf{m}/I \notin \mathrm{Ass}_R(R)$, therefore $\mathrm{Ass}_R(R) = \emptyset$ and so $Z_R(R)$ is finite union of prime ideals of R while it has not very few zero-divisors. □

REMARK 3.1.3. The following statements hold:

(1) Let R be a Noetherian local ring. If M is a balanced big Cohen-Macaulay R-module, then M has very few zero-divisors [**BH**, Proposition 8.5.5].

(2) Let R be a Noetherian ring, I an ideal of R and M a finitely generated R-module. Let the integer $i \geq 0$ be such that $H^j_I(M)$ is finitely generated for all $j < i$. Then $H^i_I(M)$ has very few zero-divisors [**BL**, Theorem 2.1].

(3) Let M be a flat R-module with $\mathrm{Ann}(M) = 0$. If R as an R-module has very few zero-divisors, then M has very few zero-divisors. It is easy to see that $Z_R(M) = Z(R)$. Thus $Z_R(M)$ is a finite union of prime ideals of $\mathrm{Ass}_R(R)$. Let $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ be maximal among those primes in $\mathrm{Ass}_R(R)$. So $Z_R(M) = \cup_{k=1}^n \mathbf{p}_k$. Now suppose that $\mathbf{p}_i = \mathrm{Ann}(x)$, where x is a nonzero element of R and $1 \leq i \leq n$. Thus $\mathbf{p}_i x = 0$. By assumption $xM \neq 0$, so there is a nonzero m in M such that $xm \neq 0$. Obviously, $\mathbf{p}_i xm = 0$, therefore, there exists an $1 \leq j \leq n$ such that $\mathbf{p}_i \subseteq \mathrm{Ann}(xm) \subseteq \mathbf{p}_j$. But according to the choice of the primes $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ we have $i = j$ and $\mathbf{p}_i = \mathrm{Ann}(xm)$ and so $\mathbf{p}_i \in \mathrm{Ass}_R(M)$.

(4) Let M be an R-module and S a multiplicatively closed subset of R such that $S \subseteq R - Z_R(M)$. Then M as an R-module has very few zero-divisors if and only if M_S as an R_S-module has very few zero-divisors.

(5) If M_i as an R_i-module, for $i = 1, \ldots, n$, has very few zero-divisors, then $\oplus_{i=1}^n M_i$ as an $\oplus_{i=1}^n R_i$-module has very few zero-divisors.

(6) If M as an R-module has very few zero-divisors and S is a commutative, cancellative and torsion-free monoid, then $M[S]$ as an $R[S]$-module has very few zero-divisors [**Na**].

LEMMA 3.1.4. *Let M and N be R-modules. If* $\mathrm{Ann}_R(x) \nsubseteq Z_R(M)$*, for any nonzero $x \in N$, then* $\mathrm{Hom}_{R_{\boldsymbol{p}}}(N_{\boldsymbol{p}}, M_{\boldsymbol{p}}) = 0$ *for all* $\boldsymbol{p} \in \mathrm{Spec}(R)$.

PROOF. Suppose $f : N_{\mathbf{p}} \longrightarrow M_{\mathbf{p}}$ is an $R_{\mathbf{p}}$-homomorphism and x/s is a nonzero element of $N_{\mathbf{p}}$. By assumption, there exists an $r \in R - Z_R(M)$ such that $rx = 0$. It is easy to check that $r/1 \notin Z_{R_{\mathbf{p}}}(M_{\mathbf{p}})$. Now we have $(r/1)f(x/s) = f(rx/s) = 0$ and therefore $f(x/s) = 0$. This implies that $\mathrm{Hom}_{R_{\mathbf{p}}}(N_{\mathbf{p}}, M_{\mathbf{p}}) = 0$. □

THEOREM 3.1.5. *Let the R-module M have very few zero-divisors and N be a locally Nakayama R-module. Then there exists a nonzero $x \in N$ such that* $\mathrm{Ann}(x) \subseteq Z_R(M)$ *if and only if* $\mathrm{Hom}_{R_{\boldsymbol{p}}}(N_{\boldsymbol{p}}, M_{\boldsymbol{p}}) \neq 0$ *for some* $\boldsymbol{p} \in \mathrm{Spec}(R)$.

PROOF. Suppose the R-module M has very few zero-divisors and there exists an $x \in N - \{0\}$ such that $\mathrm{Ann}(x) \subseteq Z_R(M)$. By the Prime Avoidance Theorem, there exists a prime ideal $\mathbf{p}$ in $\mathrm{Ass}_R(M)$ such that $\mathrm{Ann}(x) \subseteq \mathbf{p}$, and therefore $N_{\mathbf{p}} \neq 0$. But N is a locally Nakayama R-module, so $N_{\mathbf{p}}/\mathbf{p}N_{\mathbf{p}} \neq 0$ and there is a nonzero epimorphism $f : N_{\mathbf{p}} \longrightarrow k(\mathbf{p})$. Also $R/\mathbf{p} \longrightarrow M$ and consequently $g : k(\mathbf{p}) \longrightarrow M_{\mathbf{p}}$ is a nonzero monomorphism. Therefore $g \circ f \neq 0$ and consequently $\mathrm{Hom}_{R_{\mathbf{p}}}(N_{\mathbf{p}}, M_{\mathbf{p}}) \neq 0$. The converse is obviously true by Lemma 3.1.4. □

COROLLARY 3.1.6. *Let the R-module M have very few zero-divisors and N be a finitely generated R-module. Then the following statements are equivalent:*

(1) $\mathrm{Hom}(N, M) \neq 0$*;*
(2) $\mathrm{Ann}(N) \subseteq Z_R(M)$*;*
(3) $\mathrm{Supp}(N) \cap \mathrm{Ass}(M) \neq \emptyset$.

PROOF. (1) $\to$ (2) If $\mathrm{Ann}_R(N) \not\subseteq Z_R(M)$, then it is easy to see that $\mathrm{Hom}_R(N, M) = 0$ which is a contradiction, so that $\mathrm{Ann}_R(N) \subseteq Z_R(M)$.

(2) $\to$ (3) By assumption M has very few zero-divisors, so there exists $\mathbf{p} \in \mathrm{Ass}_R(M)$ such that $\mathrm{Ann}_R(N) \subseteq \mathbf{p}$. Thus $\mathrm{Supp}(N) \cap \mathrm{Ass}(M) \neq \emptyset$.

(3) $\to$ (1) By assumption there exists $\mathbf{p} \in \mathrm{Ass}_R(M)$ such that $\mathrm{Ann}_R(N) \subseteq \mathbf{p}$. Since M has a submodule isomorphic to $R/\mathbf{p}$ and $N/\mathbf{p}N$ is a quotient module of N, it is enough to show that $\mathrm{Hom}_R(N/\mathbf{p}N, R/\mathbf{p}) \neq 0$. But, $N/\mathbf{p}N$ and $R/\mathbf{p}$ both have natural structures as $R/\mathbf{p}$-modules, and any mapping $N/\mathbf{p}N \longrightarrow R/\mathbf{p}$ is an R-homomorphism if and only if it is an $R/\mathbf{p}$-homomorphism. It is, therefore, enough to show that $\mathrm{Hom}_{R/\mathbf{p}}(N/\mathbf{p}N, R/\mathbf{p}) \neq 0$. Now, let R be an integral domain and N a finitely generated R-module with $\mathrm{Ann}_R(N) = 0$. We show that $\mathrm{Hom}_R(N, R) \neq 0$. By assumption (0) is a element of $\mathrm{Supp}(N)$, thus there exists a nonzero element of N, say n, such that $\mathrm{Ann}_R(n) = 0$. Hence the assignment $r \longrightarrow rn$ induces a monomorphism $f : R \longrightarrow N$. On the other hand, $K = R_{(0)}$, the quotient field of R, is an injective K-module and so it is an injective R-module. Hence there is a nonzero R-homomorphism $g : N \longrightarrow K$ such that $g \circ f$ is inclusion map from R to K. Since N is a finitely generated R-module thus there is $s \in R - \{0\}$ such that $sg : N \longrightarrow R$ is a nonzero R-homomorphism and so $\mathrm{Hom}_R(N, R) \neq 0$. □

3.2. Some homological properties of ZD-modules

Recall that R is a commutative ring with identity. Let M be an R-module. We say that $a \in R$ is an *M-regular* element if $am \neq 0$ for all $0 \neq m \in M$, in other words, a is not a zero-divisor on M. A sequence $\mathbf{a} = a_1, \ldots, a_n$ of elements of R is called an *M-sequence*, if the following conditions are satisfied:

(1) a_i is an $M/(a_1, \ldots, a_{i-1})M$-regular element for $i = 1, \ldots, n$, and (2) $M/\mathbf{a}M \neq 0$. Note that if $n = 0$, then $(a_1, \ldots, a_n) = 0$.

Let I be an ideal of R generated by $\mathbf{x} = x_1, \ldots, x_n$. If all the Koszul homology modules $H_i(\mathbf{x}, M)$ vanish, then we set $\mathrm{grade}(I, M) = \infty$; otherwise $\mathrm{grade}(I, M) = n - h$ where $h = \sup\{i : H_i(\mathbf{x}, M) \neq 0\}$ [**BH**, Definition 9.1.1].

THEOREM 3.2.1. *Let I be a finitely generated ideal of R and R-module M have very few zero-divisors. Then the following statements are equivalent:*

(1) $I \subseteq Z_R(M)$*;*
(2) $\mathrm{Hom}_R(R/I, M) \neq 0$*;*
(3) $\mathrm{grade}(I, M) = 0$.
(4) $\Gamma_I(M) \neq 0$*, where $\Gamma_I(M)$ denotes the I-torsion functor.*

PROOF. (1) $\Leftrightarrow$ (2) is obvious by Corollary 3.1.6, and (2) $\Leftrightarrow$ (3) follows by [**BH**, Proposition 9.1.2]. The proof of (4) $\Leftrightarrow$ (1) is similar to [**BS**, Lemma 2.1.1 (ii)]. □

THEOREM 3.2.2. *Let R be a commutative ring (not necessarily Noetherian), I a finitely generated ideal of R and M an R-module such that $M \neq IM$ and $M/\mathbf{x}M$ have very few zero-divisors for any ideal $\mathbf{x}$ of R generated by an M-sequence in I. For a given integer $n > 0$ the following conditions are equivalent:*

(1) $\mathrm{Ext}^i_R(R/I, M) = 0$ *for all $i < n$;*
(2) *there exists an M-sequence of length n contained in I.*

PROOF. (1) $\to$ (2) Since $\mathrm{Hom}_R(R/I, M) = 0$, then $I \not\subseteq Z_R(M)$ by Theorem 3.2.1. So I contains an M-regular element x_1 such that $M \neq x_1 M$, since $M \neq IM$, now if $n = 1$, then we are done. If $n > 1$ we set $M_1 = M/x_1 M$. Consider that $M_1 \neq IM_1$ and $M_1/\mathbf{x}M_1$ has very few zero-divisors for any ideal $\mathbf{x}$ of R generated by an M_1-sequence in I. Since the following sequence is exact

$$0 \to M \xrightarrow{x_1} M \to M_1 \to 0$$

we get $\mathrm{Ext}^i_R(R/I, M_1) = 0$ for all $i < n-1$, so that by induction I contains an M_1-sequence $x_2, \ldots, x_n$. Thus $x_1, x_2, \ldots, x_n$ is an M-sequence in I.

Notice that for proving (2) $\to$ (1), we do not need to assume that $M/\mathbf{x}M$ has very few zero-divisors, for any ideal $\mathbf{x}$ of R generated by an M-sequence in I and the proof is similar to the proof of [**Ma**, Theorem 16.6]. □

THEOREM 3.2.3. *Let R be a commutative ring (not necessarily Noetherian), I a finitely generated ideal of R, and M an R-module such that $M \neq IM$ and $M/\boldsymbol{x}M$ have very few zero-divisors for any ideal $\boldsymbol{x}$ of R generated by an M-sequence in I; then the length of a maximal M-sequence in I is a well-determined integer n, and n is determined by* $\mathrm{Ext}^i_R(R/I, M) = 0$ *for $i < n$ and* $\mathrm{Ext}^n_R(R/I, M) \neq 0$. *Furthermore $n =$* $\mathrm{grade}(I, M)$ *defined by the concept of Koszul complex in* [**BH**, Definition 9.1.1].

PROOF. The proof of well-determinedness of the length of a maximal M-sequence in I is nothing but the mimic of the proof offered in [**Ma**, p. 130].

Now let $\mathbf{y} = y_1, \ldots, y_n$ be a maximal M-sequence in $I = (x_1, \ldots, x_k) = (\mathbf{x})$. Then $H_{k+1-i}(\mathbf{x}, M) = 0$ for $i = 1, \ldots, n$, and $H_{k-n}(\mathbf{x}, M) \cong \mathrm{Hom}_R(R/I, M/\mathbf{y}M) \cong \mathrm{Ext}_R^n(R/I, M)$ by [**BH**, Theorem 1.6.16]. Note that by $H_i(\mathbf{x}, M)$, we mean the i-th Koszul homology module of M with respect to $\mathbf{x}$. By assumption $M/\mathbf{y}M$ has very few zero-divisors and $I \subseteq Z_R(M/\mathbf{y}M)$. Therefore $H_{k-n}(\mathbf{x}, M) \cong \mathrm{Hom}_R(R/I, M/\mathbf{y}M) \neq 0$ by Theorem 3.2.1 and consequently $n = \mathrm{grade}(I, M)$. □

The i-th local cohomology module of M with respect to I is defined as

$$H_I^i(M) = \varinjlim_{n \in \mathbb{N}} \mathrm{Ext}_R^i(R/I^n, M).$$

The reader can refer to [**BS**], for the basic properties of local cohomology module. Now we offer a generalization of an important theorem in theory of local cohomology.

THEOREM 3.2.4. *Let R be a commutative ring (not necessarily Noetherian), I a finitely generated ideal of R, and M an R-module such that $M \neq IM$ and $M/\mathbf{x}M$ have very few zero-divisors for any ideal $\mathbf{x}$ of R generated by an M-sequence in I. Then* $\mathrm{grade}(I, M)$ *is the least integer i such that $H_I^i(M) \neq 0$.*

PROOF. Let $n = \mathrm{grade}(I, M)$. We use induction on n. For $n = 0$, Theorem 3.2.1 shows that $\Gamma_I(M) \neq 0$. Now assume that $n > 0$ and that the result has been proved for R-module N with $N \neq IN$, $N/\mathbf{x}N$ has very few zero-divisors for any ideal $\mathbf{x}$ of R generated by an N-sequence in I and $\mathrm{grade}(I, N) < n$. There exists $x_1 \in I$ such that x_1 is M-regular. Set $M_1 = M/x_1M$. Observe that $M_1 \neq IM_1$, $M_1/\mathbf{x}M_1$ has very few zero-divisors for any ideal $\mathbf{x}$ of R generated by an M_1-sequence in I and $\mathrm{grade}(I, M_1) = n-1$ by Proposition 9.1.2 (b) in [**BH**]. Therefore by the inductive hypothesis, $H_I^i(M_1) = 0$ for all $i < n-1$, while $H_I^{n-1}(M_1) \neq 0$. The exact sequence $0 \to M \xrightarrow{x_1} M \to M_1 \to 0$ induces, for each $i > 0$, an exact sequence

$$H_I^{i-1}(M) \to H_I^{i-1}(M_1) \to H_I^i(M) \xrightarrow{x_1} H_I^i(M).$$

This shows that, for $i < n$, the element x_1 is a non-zero divisor on $H_I^i(M)$, so that, since this module is I-torsion, it must be zero. We therefore have an exact sequence $0 \to H_I^{n-1}(M_1) \to H_I^n(M)$, and since $H_I^{n-1}(M_1) \neq 0$, it follows that $H_I^n(M) \neq 0$. □

COROLLARY 3.2.5. *Let R be a commutative Noetherian ring, I an ideal of R, and M an R-module such that $M \neq IM$ and $M/\mathbf{x}M$ have very few zero-divisors for any ideal $\mathbf{x}$ of R generated by an M-sequence in I. Then* $\mathrm{grade}(I, M) \leq \dim M$, *where* $\dim M$ *is the Krull dimension of M.*

PROOF. This is immediate from Grothendieck's Vanishing Theorem [**BS**, Theorem 6.1.2], and Theorem 3.2.4. □

A good class of modules which satisfies the condition "$M/\mathbf{x}M$ has very few zero-divisors for any ideal $\mathbf{x}$ of R generated by an M-sequence in I" in Theorem 3.2.2, 3.2.3 and 3.2.4 and Corollary 3.2.5 is the ZD-modules introduced in [**DE**]. Example 2.2 in [**DE**] shows that the class of ZD-modules includes all finitely generated, Artinian and Matlis reflexive modules. In our terminology, an R-module M is a ZD-module, if M/N has very few zero-divisors, for any submodule N of M. In the following, we show that how one can get a module having very few zero-divisors from a ZD-module. In fact, the following result is a generalization of a theorem proved in [**BL**].

THEOREM 3.2.6. *Let R be a commutative Noetherian ring, I an ideal of R, and M a ZD-module. Then, for all submodules N of M, the following statements are equivalent:*

(1) $\mathrm{Ass}_R(H_I^0(M/N)/L)$ *is finite for all finitely generated submodules L of $H_I^0(M/N)$;*
(2) *if $H_I^0(M/N),\ldots,H_I^{i-1}(M/N)$ are finitely generated, then* $\mathrm{Ass}_R(H_I^i(M/N)/L)$ *is finite for all finitely generated submodules L of $H_I^i(M/N)$.*

PROOF. It is clear that (2) implies (1). The proof of $(1)\to(2)$, which we include for the reader's convenience, is based on the proof of Proposition 2.1 in [**BL**]. We use induction on i. The claim for $i=0$ holds by the assumption. Assume that $i>0$ and the result has been proved for each ZD-module which satisfies the condition (2) for $j<i$. Let N be a submodule of M and $X=M/N$. By [**BS**, Corollary 2.1.7], since $H_I^k(X)=H_I^k(X/\Gamma_I(X))$ for all $k>0$, we may assume that X is I-torsion free. We use Theorem 3.2.1 to deduce that I contains a element r which is not a zero-divisor on X. By our choice of L, there is some $n>0$ with $r^nL=0$. We set $a=r^n$ and apply cohomology to the exact sequence $0\to X\xrightarrow{a}X\to X_1\to 0$, where $X_1=X/aX$. It follows that $H_I^l(X_1)$ is finitely generated for all $l<i-1$. Since X_1 is a ZD-module, by induction $\mathrm{Ass}_R(H_I^{i-1}(X_1)/K)$ is finite for all finitely generated submodule K of $H_I^{i-1}(X_1)$. Moreover, we get the following exact sequences in which Δ is the connecting homomorphism and φ is the natural map:

$$H_I^{i-1}(X)\xrightarrow{\varphi}H_I^{i-1}(X_1)\xrightarrow{\Delta}H_I^i(X)\xrightarrow{a}H_I^i(X).$$

As $\mathrm{Ker}(\Delta)=\varphi(H_I^{i-1}(X))$ and L are both finitely generated, so is $\Delta^{-1}(L)$. Therefore $Y=H_I^{i-1}(X_1)/\Delta^{-1}(L)$ has only finitely many associated primes. It is enough to show that

$$\mathrm{Ass}_R(H_I^i(X)/L)\subseteq\mathrm{Ass}_R(Y)\cup\mathrm{Ass}_R(L).$$

Let $\mathbf{p}\in\mathrm{Ass}_R(H_I^i(X)/L)-\mathrm{Ass}_R(Y)$. Then $\mathbf{p}=\mathrm{Ann}_R(h+L)$ for some $h\in H_I^i(X)$. The following exact sequence

$$0\to H_I^{i-1}(X_1)/\Delta^{-1}(L)\xrightarrow{\overline{\Delta}}H_I^i(X)/L\xrightarrow{\overline{a}}H_I^i(X)$$

and that $\mathbf{p}$ is not an associated prime ideal of Y and all of its submodules, show that $\mathbf{p} \in \mathrm{Ass}_R(Rah)$. Then there exists an $s \in R$ such that $\mathbf{p} = \mathrm{Ann}_R(sah)$. Since sah is annihilated by some power of a, we have $a \in \mathbf{p}$. Therefore $a(h+L) = L$ and this means that $sah \in L$ which implies that $\mathbf{p} \in \mathrm{Ass}_R(L)$. This proves the inclusion and thus our result. $\square$

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