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M-cancellation Ideals

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Let R be a commutative ring with non–zero identy and let M be an R–module. An ideal $\mathfrak{a}$ of R is called an M-cancellation ideal if whenever $\mathfrak{a}P = \mathfrak{a}Q$ for submodules P and Q of M, then $P = Q$. This notion is a generalization of the notion, cancellation ideal. We use M-cancellation ideals and a generalization of Dedekind–Mertens lemma to prove that for an R–module M with $Z_R(M) = \{0\}$, the following statements are equivalent :

(i) Every non–zero finitely generated ideal of R is an M-cancellation ideal of R.

(ii) For every $f \in R[t]$ and $g \in M[t]$, $\mathrm{c}(fg) = \mathrm{c}(f)\mathrm{c}(g)$.

1. Introduction

Let R be a commutative ring with identity. An ideal $\mathfrak{a}$ of R is called a cancellation ideal if whenever $\mathfrak{a}\mathfrak{b} = \mathfrak{a}\mathfrak{c}$ for ideals $\mathfrak{b}$ and $\mathfrak{c}$ of R, then $\mathfrak{b} = \mathfrak{c}$. A good introduction to cancellation ideals may be found in Gilmer [[3] ; section 6]. An R–module M is called cancellation module, if whenever $\mathfrak{a}M = \mathfrak{b}M$ for ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R, then $\mathfrak{a} = \mathfrak{b}$, cf. [6]. The ideal $\mathfrak{a}$ of R is called an M-cancellation ideal if whenever $\mathfrak{a}P = \mathfrak{a}Q$ for submodules P and Q of M, then $P = Q$. In the first section we characterize M-cancellation ideals. Let M be a cancellation module and let $\mathfrak{a}$ be an ideal of R. Then $\mathfrak{a}$ is an M-cancellation ideal of R if and only if $\mathfrak{a}$ is locally an M–regular principal ideal of R. This result is a generalization of Anderson and Roitman in [2].

In section 2 we use M-cancellation ideals and a generalization of Dedekind–Mertens lemma to prove some results of content formulas for polynomial modules such as, for an R–module M with $Z_R(M) = \{0\}$, the following statements are equivalent:

(i) Every non–zero finitely generated ideal of R is an M-cancellation ideal of R.

(ii) For every $f \in R[t]$ and $g \in M[t]$, $\mathrm{c}(fg) = \mathrm{c}(f)\mathrm{c}(g)$.

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2. M-cancellation Ideals

An ideal $\mathfrak{a}$ of a ring R is called a cancellation ideal if for all ideals $\mathfrak{b}$ and $\mathfrak{c}$ of R, $\mathfrak{a}\mathfrak{b} = \mathfrak{a}\mathfrak{c}$ implies $\mathfrak{b} = \mathfrak{c}$. In [2], Anderson and Roitman showed that the ideal $\mathfrak{a}$ of R is a cancellation ideal if and only if $\mathfrak{a}$ is locally a regular principal ideal. As a generalization of cancellation ideal the R–module M is called a cancellation module if for all ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R, $\mathfrak{a}M = \mathfrak{b}M$ implies $\mathfrak{a} = \mathfrak{b}$.

Definition 2.1. Let M be an R–module. An ideal $\mathfrak{a}$ is called an M-cancellation ideal if for all submodules P and Q of M, $\mathfrak{a}P = \mathfrak{a}Q$ implies $P = Q$.

The following lemmas are straightforward, but we write them here only for the sake of references.

Lemma 2.2. *Let M be an R–module and let $\mathfrak{a}$ be an ideal. If M is a cancellation module and $\mathfrak{a}$ is an M-cancellation ideal of R then $\mathfrak{a}$ is a cancellation ideal.*

Lemma 2.3. *Let M be an R–module and $x \in R$. The ideal $\mathfrak{a} =< x >$ is an M-cancellation ideal if and only if $x \notin Z_R(M)$.*

Lemma 2.4. *Let M be an R–module, and let $\mathfrak{a}$ be an ideal of R. If $\mathfrak{a}$ is locally an M-cancellation ideal of R (i.e. for every maximal ideal $\mathfrak{m}$ of R, $\mathfrak{a}_\mathfrak{m}$ is an $M_\mathfrak{m}$–cancellation ideal of $R_\mathfrak{m}$) then $\mathfrak{a}$ is an M-cancellation ideal of R.*

Now we prove that the converse of lemma 2.4 is correct when M is a cancellation module.

Theorem 2.5. *Let M be a cancellation R–module and let $\mathfrak{a}$ be an ideal of R. The following statements are equivalent* :

(i) $\mathfrak{a}$ *is an M-cancellation ideal of R.*

(ii) $\mathfrak{a}$ *is locally an M–regular principal ideal of R.*

(iii) $\mathfrak{a}$ *is locally an M-cancellation ideal of R.*

Proof. (ii)⇒(iii)⇒(i) are straightforward.

(i)⇒(ii): Suppose that $\mathfrak{a}$ is an M-cancellation ideal of R, by lemma 2.2 the ideal $\mathfrak{a}$ is a cancellation ideal of R and so $\mathfrak{a}$ is locally principal, cf. [2]. Let $\mathfrak{m}$ be a maximal ideal of R and let $\mathfrak{a}_\mathfrak{m} =< x >_\mathfrak{m}$ where $x \in R$. We claim that $x/1$ is $M_\mathfrak{m}$–regular. Suppose that $y/1 \in M_\mathfrak{m}$ and $xy = 0$ in $M_\mathfrak{m}$. Then $(\mathfrak{a}y)_\mathfrak{m} = (xy)_\mathfrak{m} = 0$. Since $(\mathfrak{a}y)_\mathfrak{n} = (\mathfrak{m}\mathfrak{a}y)_\mathfrak{n}$ for all other maximal ideal $\mathfrak{n}$ of R, we have $\mathfrak{a}y = \mathfrak{m}\mathfrak{a}y$. Since $\mathfrak{a}$ is an M-cancellation ideal of R, $< y >= \mathfrak{m} < y >$. Then $y = 0$ in $M_\mathfrak{m}$.

3. Content Formulas for Polynomial Modules

Let M be an R–module. For $g \in M[t]$, the content $\mathrm{c}(g)$ of g is the R–module of N generated by the coefficients of g.

In [[1] ; Remark 1.5] Anderson and Kang proved the following theorem;

Theorem 3.1. *Suppose $f \in R[t]$. Then the following are equivalent :*

(i) *$c(f)$ is locally principal ideal of R.*

(ii) *For all R–module M and $g \in M[t]$, $c(fg) = c(f)c(g)$.*

Now let M be an R–module and $g \in M[t]$ such that $\mathrm{c}(g)$ is locally a cyclic submodule of M, we prove for all $f \in R[t]$, $\mathrm{c}(fg) = \mathrm{c}(f)\mathrm{c}(g)$. Before proving the above statement we state the following lemma.

Lemma 3.2. *Let M be an R–module and $g \in M[t]$ such that $c(g) =< m >$ for $m \in M$. There exists $g_1 \in R[t]$ such that $g = g_1 m$ and $c(g_1) = R$.*

Proof. Suppose that $g = m_0 + m_1 t + \cdots + m_n t^n$ and $\mathrm{c}(g) =< m_0, \ldots, m_n >=< m >$, so we have $m = \sum_{i=0}^{n} r_i m_i$ and $m_i = s_i m$, so if we suppose that $d = \sum_{i=0}^{n} r_i s_i$ we have $(1-d)m = 0$. Put $g_1 = s_0 + s_1 t + \cdots + s_n t^n + (1-d)t^{n+1}$. It is easy to see that $g = g_1 m$ and $\mathrm{c}(g_1) = R$.

Theorem 3.3. *Let M be an R–module and $g \in M[t]$ such that $c(g)$ is locally a cyclic submodule of M. Then for all $f \in R[t]$, $c(fg) = c(f)c(g)$.*

Proof. We may reduce to the case $(R, \mathfrak{m})$ is a local ring and $\mathrm{c}(g) =< m >$ where $g \in M[t]$ and $m \in M$. By lemma 2.2 there is a $g_1 \in R[t]$ such that $g = g_1 m$ and $\mathrm{c}(g_1) = R$. Now we see that

$$\mathrm{c}(fg) = \mathrm{c}(fg_1 m) = \mathrm{c}(fg_1)m = \mathrm{c}(f)m = \mathrm{c}(f)\mathrm{c}(g)$$

Definition 3.4. Let M be an R–module. The M–Dedekind–Mertens number $\mu(g)$ of a polynomial $g \in M[t]$ is the smallest positive integer k such that

$$\mathrm{c}(f)^{k-1}\mathrm{c}(f)\mathrm{c}(g) = \mathrm{c}(f)^{k-1}\mathrm{c}(fg)$$

for every polynomial $f \in R[t]$. The polynomial $g \in M[t]$ is said to be an M–Gaussian polynomial if $\mu(g) = 1$.

Definition 3.5. Let M be an R–module and let N be a submodule of M. Then $\mu(N)$ denotes the minimal number of generators of N.

The next theorem is a generalization of [[4] ; Theorem 2.1]. The proof of the theorem (3.6) is the same as the proof of [[4] ; Theorem 2.1] and so we omit it.

Theorem 3.6. *Let M be an R–module and $g \in M[t]$. If for each maximal ideal $\mathfrak{m}$ of R, $c(g)_{\mathfrak{m}}$ is generated in $M_{\mathfrak{m}}$ by k elements, then the M–Dedekind–Mertens number $\mu(g) \le k$, i.e. for every polynomial $f \in R[t]$, $c(f)^k c(g) = c(f)^{k-1} c(fg)$.*

Corollary 3.7. *Let M be an R–module and let $f \in R[t]$. If the content ideal of f, $c(f)$ is an M-cancellation ideal of R, then for every $g \in M[t]$ we have $c(fg) = c(f)c(g)$.*

Lemma 3.8. *Let M be an R–module. Then*

(a) *If I, J and K are ideals of R such that $I+J$, $J+K$ and $K+I$ are M–cacellation ideals then $I+J+K$ is M-cacallation ideal of R.*

(b) *If every principal and 2–generated ideal of R are M-cancellation ideals then every finitely generated ideal of R is an M-cancellation ideal.*

Proof. (a) : The assertion follows from the fact

$$(I+J)(J+K)(K+I) = (I+J+K)(IJ+JK+KI).$$

(b) : We prove (b) by induction on the number of generators of an ideal. So suppose that every ideal generated by less than n elements is an M-cancellation ideal and suppose that $A =< a_1, a_2, \ldots, a_n >$ is an ideal of R. Put $I =< a_1 >$, $J =< a_2, a_3, \ldots, a_{n-1} >$ and $K =< a_n >$. By the hypothesis of induction $I+J$, $J+K$ and $K+I$ are M-cancellation ideals of R. So A is an M-cancellation ideal.

Corollary 2.7 says that if every finitely generated ideal of R is an M-cancellation ideal then for every $f \in R[t]$ and $g \in M[t]$ we have $\mathrm{c}(fg) = \mathrm{c}(f)\mathrm{c}(g)$. We prove that for some cases, the inverse is correct too.

Theorem 3.9. *Let M be a non–zero R–module and $Z_R(M) = \{0\}$. The following statements are equivalent :*

(i) *Every non–zero finitely generated ideal of R is an M-cancellation ideal of R.*

(ii) *For every $f \in R[t]$ and $g \in M[t]$, $c(fg) = c(f)c(g)$.*

Proof. (i)$\Rightarrow$ (ii) : This follows from corollary 3.7.

(ii)$\Rightarrow$ (i) : By lemma 3.8, it is sufficient to prove that every 2-generated ideal of R is an M-cancellation ideal. Note that $\mathrm{Z}_R(M) = \{0\}$ implies that every non–zero principal ideal of R is an M-cancellation ideal. Let $x \in M$ be a non–zero element and let $\varphi : R \to M$ be the R–homomorphism defined by $\varphi(r) = rx$ for all $r \in R$. Then φ is injective.

Let $I =< a, b >$ and $f = a + bt$ and $g = \varphi(a) - \varphi(b)t$. Note that for every $r \in R$, $\varphi(r) = r\varphi(1)$ and $< \varphi(1) >= \varphi(R)$. Since $fg = a^2\varphi(1) - b^2\varphi(1)t^2$ then

$$\mathrm{c}(fg) =< a^2, b^2 >< \varphi(1) >=< a^2, b^2 > \varphi(R) = \varphi(< a^2, b^2 >).$$

Also $\mathrm{c}(fg) = \mathrm{c}(f)\mathrm{c}(g) = \varphi < a, b >^2$, so $\varphi < a^2, b^2 >= \varphi < a, b >^2$. Since φ is injective we have that $< a^2, b^2 >=< a, b >^2$. Therefore $ab \in < a^2, b^2 >$ and there exist $r, s \in R$ such that $ab = ra^2 + sb^2$. Now put $g = s\varphi(b) + r\varphi(a)t$. Then we have

$$fg = abs\varphi(1) + (ra^2 + sb^2)\varphi(1)t + abr\varphi(1)t^2 = abs\varphi(1) + ab\varphi(1)t + abr\varphi(1)t^2$$

Therefore $\mathrm{c}(fg) = \varphi < ab >$ and by a similar way, we have $< a, b >< sb, ra >=< ab >$. In fact R is an integral domain and $ab \neq 0$, so $< a, b >$ is a factor of the M-cancellation ideal $< ab >$ of R, so every 2–generated ideal of R is an M-cancellation ideal of R and this completes the proof.

Now we give some examples that satisfy the conditions of theorem 3.9.

Example 3.10.

(i) Let D be an integral domain and M be a free D–module.

(ii) Let S be an integral domain and let R be a subring of S and regard S as an R–module. The interesting case is when R is an integral domain and S is its quotient field.

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References

[1] D. D. Anderson and B. G. Kang, *Content formulas for polynomials and power series and complete integral closure*, J. Algebra, **181**(1996), 82-94.

[2] D. D. Anderson and M. Roitman, *A characterization of cancellation ideals*, Proc. Amer. Math. Soc., **125**(1997), 2853-2854.

[3] R. Gilmer, *Multiplicative ideal theory*, Marcel Dekker, NewYork, (1972).

[4] W. Heinzer and C. Huneke, *The Dedekinds–Mertens lemma and the content of polynomials*, Proc. Amer. Math. Soc., **126**(1998), 1305-1309.

[5] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, (1986).

[6] A. G. Naoum and A. S. Mijbass , *Weak cancellation modules*, Kyungpook Math. J., **37**(1997), 73-82.